Introduction to Homogenization Theory
Applied to the Study of Composite Materials

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SECTION 1

Introduction

The need for advanced lightweight composite materials in today’s aerospace and aviation industries (to name but two) is now more pronounced than ever. In view of this trend, basic research in the physics of composite materials is of fundamental importance. This paper is an introduction to homogenization theory applied to the study of composite materials.

The overall elastic, electric, and thermal properties of composite materials have long been studied by physicists and materials scientists. Recently, a growing number of mathematicians have also been studying such effective moduli of composite materials, using the relatively new notion of $G$-convergence, or homogenization, to provide a firm mathematical base for this study ([3, 19, 34, 41, 42, 43, 46, 48]).

A composite material is a heterogeneous medium whose phases oscillate on a length-scale that is small compared to that on which the load and boundary conditions vary. The effective moduli of such a medium are defined in a manner so as to describe its overall macroscopic behavior. Mathematicians have treated three classes of microstructures (fine-scale structures) of composite media, which we list in order of increasing generality.

The first is the microstructure of a periodic composite, that is, one with a specified unit cell. The second and more general microstructure is that of a random composite, which is a stochastic process with specified statistics. The third and most general microstructure is one for which neither of the above hypotheses is imposed on the composite; here we appeal to a compactness theorem of partial differential equations known as $G$-convergence (convergence of Green’s functions) or homogenization. The random as well as the most general $G$-convergent cases lie outside the scope of this paper.

In this paper, we treat the case of periodic microstructures. It turns out that all the results obtained for periodic microstructures apply to the more general random and $G$-convergent microstructures (see [13, 21]). To fix ideas, we shall consider periodic composites made of two isotropic materials mixed in some specified proportions; in fact, we shall take as our setting the study of electricity. This paper is outlined as follows.
In Section 2, we state the definition of the effective conductivity tensor of a periodic two-phase composite, given by the variational principle (2.1). Since, in general, the effective electrical conductivity of a material is represented by a symmetric positive definite matrix, the set of effective conductivity tensors of periodic two-phase composites is characterized by bounds on their eigenvalues.

We state two pairs of eigenvalue bounds on this set of effective conductivity tensors. First, we state the well-known harmonic mean-arithmetic mean bounds (2.2), which were developed early in this century ([15, 16, 37, 44, 47]). Second, we state the trace bounds (2.5)–(2.6) which correlate the eigenvalues of the effective tensor. The trace bounds are relatively new, and were developed by Tartar and Murat, and independently by Lurie and Cherkaev, using the method of compensated compactness ([24, 27, 45]). These bounds are independent of the geometry of the composite and, together, prove to be optimal or attainable; i.e., for any positive symmetric matrix whose eigenvalues satisfy these bounds, one can construct a composite material whose effective conductivity tensor is given precisely by that matrix. This last fact is stated in Theorem 2.1. We mention that it was noticed by Kohn and Milton [22] that the trace bounds can also be derived using the Hashin-Shtrikman variational principles [14]. Recently, Milton [32] has established a link between the Hashin-Shtrikman variational principles and the compensated compactness method, and has shown them both to be special cases of a general principle that he calls the translation method.

In Section 3, we motivate the definition of the effective conductivity tensor of a periodic two-phase composite material by physical considerations; namely, we equate the total amount of heat energy produced by an applied (electric) potential field in a periodic two-phase conductor to that of an “effective” homogeneous conductor. We take this avenue because we wish to treat any periodic two-phase material as an “equivalent” homogeneous, possibly anisotropic one. To fix ideas, we treat the 2-dimensional case here; the $n$-dimensional case is handled similarly.

In Section 4, we introduce the function spaces that we shall work in throughout the rest of this paper. Next, we introduce the notion of $G$-convergence of elliptic operators. Then, using the language of $G$-convergence, we state the Fundamental Theorem of Periodic Homogenization. It is the Fundamental Theorem that provides the rigorous mathematical setting for the notion of the effective conductivity of a periodic two-phase composite that was motivated in Section 3 by purely physical considerations.

In Section 5, we motivate the Fundamental Theorem via a formal two-scale asymptotic analysis.
In Section 6, we prove the Fundamental Theorem using the “energy method” due to Tartar [5, 39].

In Section 7, we establish the bounds on the set of effective conductivity tensors of periodic two-phase composites stated in Section 2. We establish the classical harmonic mean-arithmetic mean bounds (given here in (7.2)). Following Kohn and Milton [22] we then derive the Hashin-Shtrikman variational principles which we use, instead of the method of compensated compactness, to establish the trace bounds (given here in (7.3)–(7.4)); the Hashin-Shtrikman variational principles are derived directly from the definition of the effective conductivity tensor of a periodic two-phase composite (7.1).

In Section 8, we prove Theorem 2.1; i.e., we prove the attainability of the bounds (7.2)–(7.4), as well as all points lying within these bounds. Indeed, we prove the Theorem in two parts: first, we show in Lemma 8.1 that the trace bounds (7.3)–(7.4) are attainable subject to the harmonic mean-arithmetic mean bounds (7.2); and then we show in Lemma 8.2 that all points (eigenvalues) lying within these bounds are also attainable. Our composites achieving these bounds, and all points in between, are constructed using the well-known successive laminate construction ([1, 2, 9, 10, 11, 20], [23]–[31], [40, 45]). To fix ideas, we again consider the 2-dimensional case, and remark that the n-dimensional case is handled analogously.
The setup, the results

Consider a mixture of two isotropic materials with electric conductivities $\alpha$ and $\beta$, $0 < \alpha < \beta < \infty$, in some specified proportions $\Theta_\alpha$ and $\Theta_\beta = 1 - \Theta_\alpha$; we shall refer to the constituents as the $\alpha$- and $\beta$-materials, respectively. Denoting a period cell by $Q \subseteq \mathbb{R}^n$, define the $Q$-periodic function $a(y)$ by

$$a(y) = \alpha \chi_\alpha(y) + \beta \chi_\beta(y),$$

where $\chi_\alpha$ and $\chi_\beta$ are the characteristic functions of the $\alpha$- and $\beta$-materials; i.e.,

$$\chi_\alpha(y) = \begin{cases} 1, & \text{in the } \alpha\text{-material} \\ 0, & \text{otherwise} \end{cases},$$

$$\chi_\beta(y) = 1 - \chi_\alpha(y).$$

Evidently, the volume fractions of the $\alpha$- and $\beta$-materials are given by

$$\Theta_\alpha = \int_Q \chi_\alpha(y) \, dy,$$

$$\Theta_\beta = 1 - \Theta_\alpha,$$

where $\int_Q$ denotes the average over $Q$.

The effective conductivity tensor $A$ of a periodic two-phase composite is then defined by the variational principle

$$\langle A \xi, \xi \rangle = \inf_v \int_Q a(y) |\xi + \nabla v|^2 \, dy,$$

where $\xi$ is any vector in $\mathbb{R}^n$, and $v$ ranges over all $Q$-periodic functions. (See Sections 3 and 4.)

Remark 2.1. Throughout this paper, unless explicitly stated otherwise, we adhere to the following notational conventions:

$$\langle \cdot, \cdot \rangle$$ denotes the usual Euclidean inner-product, and

$$| \cdot |$$ denotes the usual Euclidean norm.
In this paper, we shall discuss two different sets of bounds on the effective conductivity tensor $A$ of a periodic two-phase composite. The first set of bounds, the harmonic mean-arithmetic mean bounds, were known early in this century:

\begin{equation}
   hI \leq A \leq mI,
\end{equation}

where

\begin{equation}
   h = \left( \int_Q a^{-1}(y) \, dy \right)^{-1} = \left( \frac{\Theta_\alpha}{\alpha} + \frac{\Theta_\beta}{\beta} \right)^{-1},
\end{equation}

\begin{equation}
   m = \int_Q a(y) \, dy = \alpha \Theta_\alpha + \beta \Theta_\beta,
\end{equation}

and $I$ is the $n \times n$ identity matrix.

These bounds were first shown using the methods that Voight [47] and Reuss [37] had suggested for calculating the parameters (such as the stiffness tensor and compliance tensor) of polycrystals, namely, by averaging the appropriate values over volume and orientation ([15, 16, 44]). Voight’s method gives the upper bound, while that of Reuss gives the lower bound.

The second (tighter) set of bounds, called the trace bounds, on (the eigenvalues of) $A$ was first obtained using the method of compensated compactness ([24, 27, 45]). The lower trace bound is

\begin{equation}
   \sum_{i=1}^n \frac{1}{\lambda_i - \alpha} \leq \frac{n-1}{m - \alpha} + \frac{1}{h - \alpha},
\end{equation}

and the upper trace bound is

\begin{equation}
   \sum_{i=1}^n \frac{1}{\beta - \lambda_i} \leq \frac{n-1}{\beta - m} + \frac{1}{\beta - h},
\end{equation}

where each $\lambda_i$ ($i = 1, 2, \ldots, n$) is an eigenvalue of $A$, and $h$ and $m$ are given by (2.3) and (2.4), respectively. These bounds were later established using representation formulae [33], and also by using the Hashin-Shtrikman variational principles [14]; a comparison of these three methods is done by Kohn and Milton [22]. We shall derive (2.5) and (2.6) in Section 7 via the Hashin-Shtrikman variational principles.

It should be noted that the bounds (2.2), (2.5), and (2.6) do not depend on the geometry of the composite, but depend only on the conductivities ($\alpha$ and $\beta$) and the volume fractions ($\Theta_\alpha$ and $\Theta_\beta$) of the composite’s constituents; these bounds are depicted in Figure 2.1 for the case $n = 2$. Moreover, these bounds completely characterize the set of effective conductivity tensors of periodic two-phase composites; indeed, we have the following theorem.
2. THE SETUP, THE RESULTS

Theorem 2.1. For any symmetric positive definite matrix $A$ satisfying the bounds (2.2), (2.5), and (2.6), one can construct a composite with effective conductivity tensor given by $A$ using two isotropic materials with conductivities $\alpha$ and $\beta$ ($0 < \alpha < \beta < \infty$) in the volume fractions $\Theta_\alpha$ and $\Theta_\beta = 1 - \Theta_\alpha$.

We shall prove Theorem 2.1 using laminar composites (Figure 2.2a) in Section 8. Such composites have been discussed by many authors, for example [1], [2], [9]–[11], [20], [23]–[31], [40], [45], and have been used to prove the attainability of many different bounds. We remark that a second construction used to prove the Theorem uses the coated ellipsoid model (Figure 2.2b).

Figure 2.2. Ranked laminar (a) and coated ellipsoid (b) models of two-phase composites.
Physical derivation of the effective conductivity tensor of a periodic two-phase composite

The key idea behind the physical derivation of the effective electrical conductivity tensor of a periodic two-phase composite is that we want ultimately to be able to treat the two-phase material as an “equivalent” homogeneous anisotropic one. We shall derive the effective conductivity tensor by equating the total amount of heat produced by an applied potential field in a periodic two-phase conductor to that of an “effective” homogeneous conductor. For concreteness, we shall restrict our discussion to the 2-dimensional case; the n-dimensional case is handled similarly.

Consider the square place $\Gamma \subseteq \mathbb{R}^2$ (Figure 3.1a) composed of a homogeneous anisotropic conductor with conductivity tensor $A$. Let $\phi$ be the potential field (assumed to be continuous) which satisfies the conditions

\begin{align}
\text{div } A \nabla \phi &= 0 \quad \text{in } \Gamma, \\
\phi &= \langle \xi, x \rangle \quad \text{on } \partial \Gamma,
\end{align}

where $\xi$ is a constant vector in $\mathbb{R}^2$. (In other words, current flows into or out of the plate via its boundary (edges) only, and the potential along the plate’s boundary is described by a linear function.)

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3_1}
\caption{A homogeneous anisotropic material (a) and a two-phase $Q$-periodic composite (b).}
\end{figure}
Notice that $\phi = \langle \xi, x \rangle$ solves (3.1) so that the total amount of heat energy produced in $\Gamma$ per unit time is

$$
\int_{\Gamma} \langle A \nabla \phi, \nabla \phi \rangle \, dx = \int_{\Gamma} \langle A \nabla \langle \xi, x \rangle, \nabla \langle \xi, x \rangle \rangle \, dx = \langle A \xi, \xi \rangle |\Gamma|,
$$

where $|\Gamma| = \int_{\Gamma} dx$.

Now, replace the homogeneous anisotropic material in $\Gamma$ with the two-phase $Q$-periodic composite shown in Figure 3.1b. The composite here is a mixture of two isotropic materials with conductivities $\alpha$ and $\beta$, $0 < \alpha < \beta < \infty$, in the proportions $\Theta_\alpha$ and $\Theta_\beta = 1 - \Theta_\alpha$.

Let $a(y)$ be the $Q$-periodic function defined by

$$
a(y) = \alpha \chi_\alpha(y) + \beta \chi_\beta(y),
$$

where $\chi_\alpha$ and $\chi_\beta$ are the characteristic functions of the $\alpha$- and $\beta$-materials. If we suppose, in this case, that the potential field $\phi$ has two components—a linear component, and a $Q$-periodic component that oscillates with the geometry (i.e., $\phi = \langle \xi, x \rangle + v$, where $v$ is a $Q$-periodic function)—then the potential field $\phi$ satisfies

$$
div a(y) \nabla \phi = 0 \quad \text{in } \Gamma,
$$

$$
\langle \alpha \nabla \phi, \bar{n} \rangle = \langle \beta \nabla \phi, \bar{n} \rangle \quad \text{on the } \alpha-\beta \text{ phase boundary},
$$

where $\bar{n}$ is the outward-unit normal vector to the $\alpha-\beta$ phase boundary.

Here we see that $\phi = \langle \xi, x \rangle + v$ solves (3.3) so that, in this case, the total amount of heat energy produced in $\Gamma$ per unit time is

$$
\int_{\Gamma} \langle a(y) \nabla \phi, \nabla \phi \rangle dy = \int_{\Gamma} \langle a(y) \nabla (\langle \xi, x \rangle + v), \nabla (\langle \xi, x \rangle + v) \rangle dy = \int_{\Gamma} a(y) |\xi + \nabla v|^2 dy.
$$

We equate the heat energies given in (3.1) and (3.4) to define the effective conductivity tensor $A$ of the periodic two-phase conductor:

$$
\langle A \xi, \xi \rangle |\Gamma| = \int_{\Gamma} a(y) |\xi + \nabla v|^2 dy,
$$
or, equivalently,

$$
\langle A \xi, \xi \rangle = \int_{\Gamma} a(y) |\xi + \nabla v|^2 dy,
$$

where $\int_{\Gamma}$ denotes the mean, i.e., we define the effective conductivity tensor $A$ of a periodic two-phase conductor to be the conductivity tensor of a homogeneous (possibly anisotropic) conductor that produces the same amount of heat energy as the periodic two-phase conductor.

Note that, from purely physical considerations, we have derived a useful (and meaningful) definition of the effective conductivity tensor of a periodic two-phase...
composite that agrees with the mathematically precise definition given by the Fundamental Theorem of Periodic Homogenization (Section 4).
The notion of $G$-convergence of elliptic operators, and the Fundamental Theorem of Periodic Homogenization

The Fundamental Theorem of Periodic Homogenization will be stated using the notion of $G$-convergence of elliptic operators; but before introducing this very useful notion, we first define the function spaces that we shall be working in throughout the rest of this paper.

Let $\Omega$ be an open, bounded region of $\mathbb{R}^n$. Then the following are inner-product spaces over the reals.

1. $L^2(\Omega) = \{ u : \Omega \to \mathbb{R} | \int_{\Omega} u^2 \, dx < \infty \}$ with inner-product defined by
   $$\langle u, v \rangle_{L^2} = \int_{\Omega} uv \, dx;$$

2. $H^1(\Omega) = \{ u \in L^2(\Omega) | \partial_{x_1} u, \partial_{x_2} u, \ldots, \partial_{x_n} u \in L^2(\Omega) \}$ with inner-product defined by
   $$\langle u, v \rangle_{H^1} = \int_{\Omega} uv \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$
   where $\cdot$ denotes the usual Euclidean inner-product in $\mathbb{R}^n$;

3. $H^1_0(\Omega) = \{ u \in H^1(\Omega) | u = 0 \text{ on } \partial \Omega \}$ with the $H^1$-inner-product.

The space $L^2(\Omega)$ is often called the space of square-integrable real-valued functions (defined on $\Omega$), while the spaces $H^1(\Omega)$ and $H^1_0(\Omega)$ are called Sobolev spaces; moreover, $H^1_0(\Omega)$ is a subspace of $H^1(\Omega)$, and $H^1(\Omega)$ is a subspace of $L^2(\Omega)$. Also, each of these three spaces is a Hilbert space with norms defined by

$$\| u \|_{L^2} = \langle u, u \rangle_{L^2}^{1/2}, \quad \| u \|_{H^1} = \langle u, u \rangle_{H^1}^{1/2}.$$ 

Another space that will be of importance to us is $C_0^\infty(\Omega)$, the space of infinitely continuously differentiable real-valued functions (defined on $\Omega$) that vanish outside closed bounded subsets of $\Omega$. $C_0^\infty(\Omega)$ is also denoted $D(\Omega)$, and is often described as the space of functions that have continuous derivatives of all orders with compact support on $\Omega$ (Figure 4.1).
Finally, we define $H^{-1}(\Omega)$ to be the dual space of $H^1_0(\Omega)$, and remark (without proof) that $L^2(\Omega) \subseteq H^{-1}(\Omega)$ (e.g. [36]).

We now introduce the notion of $G$-convergence of elliptic operators.

Let $E(\lambda_0, \Lambda_0; \Omega)$ denote the class of operators $A: H^1_0(\Omega) \to H^{-1}(\Omega)$ such that

$$A = -\partial_x (a_{ij}(x) \partial_{x_j})$$

for some real measurable functions $a_{ij}$ on $\Omega$ (called the coefficients of $A$) where

$$a_{ij} = a_{ji}, \quad \text{and}$$

$$\lambda_0 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda_0 \quad \text{for all} \quad \xi \in \mathbb{R}^n.$$  

**Remark 4.1.** We are using the Einstein summation notation, where we sum over repeated indices. So, for example,

$$a_{ij}(x)\xi_i = \sum_{i=1}^{n} a_{ij} \xi_i.$$  

Now, for $f \in H^{-1}(\Omega)$, and for $A^\varepsilon$, $A \in E(\lambda_0, \Lambda_0; \Omega)$, $0 < \lambda_0 < \Lambda_0$, where

$$A^\varepsilon = -\partial_x (a_{ij}(x/\varepsilon) \partial_{x_j}), \quad \varepsilon \in \mathbb{R},$$

consider the Dirichlet problems

$$A^\varepsilon u^\varepsilon = f \quad \text{in} \quad \Omega,$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial \Omega,$$

and

$$Au = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega.$$  

Notice that we may write the solutions $u^\varepsilon$ and $u$ of (4.1) and (4.2), respectively, as

$$u^\varepsilon = (A^\varepsilon)^{-1} f$$

and

$$u = A^{-1} f,$$

from which we see that $(A^\varepsilon)^{-1}$ and $A^{-1}$ are solution operators. This motivates the following definition.
Definition 4.1. The sequence of operators \( \{ A^\varepsilon \} \) described above \( G \)-converges (convergence of Green’s functions) to the operator \( A \) if for all \( f \) in \( H^{-1}(\Omega) \) the solutions \( u^\varepsilon \) converge to \( u \) weakly in \( H^1_0(\Omega) \) as \( \varepsilon \) tends to zero; i.e., we say that
\[
A^\varepsilon \rightharpoonup G A \quad \text{as} \quad \varepsilon \to 0
\]
for all \( f \) in \( H^{-1}(\Omega) \).

Of course, since the operators \( A^\varepsilon \) and \( A \) are determined by their coefficients \( a_{ij}(x/\varepsilon) \) and \( a_{ij}(x) \), respectively, we could just as well speak of the \( G \)-convergence of the matrices of coefficients:

Definition 4.2. The sequence of matrices \( \{ a_{ij}(x/\varepsilon) \} \) described above \( G \)-converges to the matrix \( a_{ij}(x) \) if for all \( f \) in \( H^{-1}(\Omega) \) the solutions \( u^\varepsilon \) converge to \( u \) weakly in \( H^1_0(\Omega) \) as \( \varepsilon \) tends to zero; that is, we say that
\[
a_{ij}(x/\varepsilon) \rightharpoonup G a_{ij}(x) \quad \text{as} \quad \varepsilon \to 0
\]
for all \( f \) in \( H^{-1}(\Omega) \).

We are now in a position to state the Fundamental Theorem of Periodic Homogenization using the language of \( G \)-convergence.

First, recall from our setup in Section 2 that we are considering a mixture of two isotropic materials with conductivities \( \alpha \) and \( \beta \), \( 0 < \alpha < \beta < \infty \), in some specified proportions \( \Theta_\alpha \) and \( \Theta_\beta = 1 - \Theta_\alpha \); we also defined the \( Q \)-periodic function
\[
a(x) = \alpha \chi_\alpha(x) + \beta \chi_\beta(x),
\]
where \( \chi_\alpha \) and \( \chi_\beta \) are the characteristic functions of the \( \alpha \)- and \( \beta \)-materials. Thus, at any point \( x \in \mathbb{R}^n \), the conductivity tensor of the mixture is
\[
A(x) = a(x)I,
\]
where \( a(x) \) is given by (4.3), and \( I \) is the \( n \times n \) identity matrix.

Let \( \varepsilon \) denote the length-scale of the period cells, and consider the family of mixtures \( \chi_{\alpha}^\varepsilon = \chi_{\alpha}(x/\varepsilon) \) contained in an open (Lipschitz) bounded set \( \Omega \subseteq \mathbb{R}^n \) with conductivity tensors
\[
A^\varepsilon(x) = a^\varepsilon(x)I = (\alpha \chi_{\alpha}^\varepsilon(x) + \beta \chi_{\beta}^\varepsilon(x))I.
\]

Theorem 4.2 (The Fundamental Theorem of Periodic Homogenization). As \( \varepsilon \) tends to zero, the whole sequence \( A^\varepsilon \) described above \( G \)-converges to a unique constant matrix \( A^0 \), which may be written explicitly in terms of a variational principle:
\[
\langle A^0 \xi, \xi \rangle = \inf_v \int_\Omega a(y)\| \xi + \nabla v \|^2 dy
\]
for any $\xi \in \mathbb{R}^n$, where $a(y)$ is defined by (4.3), and $v$ ranges over all $Q$-periodic functions.

Notice that Equation (4.4) is identical to Equation (2.1) on page 5. Thus, we see that the Fundamental Theorem gives the rigorous mathematical setting for the definition of the effective conductivity tensor of a periodic two-phase composite. We shall first motivate the Fundamental Theorem using the method of two-scale asymptotic expansions in Section 5, and then provide a rigorous proof of the Fundamental Theorem using the “energy method” of Tartar in Section 6. Our analysis in each of these sections is along the lines of [5, 39].
Motivating the Fundamental Theorem of Periodic Homogenization via a formal two-scale asymptotic analysis

Let $Q = (0,1)^n$, and let $a(x) \in L^2(Q)$ be the $Q$-periodic real-valued function defined by (4.3). Then extend $a(x)$ periodically to $\mathbb{R}^n$, and define

$$a^\epsilon(x)I = (a(x/\epsilon))I = (a\chi_a(x/\epsilon) + \beta\chi_b(x/\epsilon))I,$$

where $I$ is the $n \times n$ identity matrix. Next, let $\Omega$ be an open bounded region in $\mathbb{R}^n$, and consider the Dirichlet problem

$$-\partial_{x_i}(a^\epsilon(x)\partial_{x_i} u^\epsilon) = f(x) \text{ in } \Omega,$$

$$u^\epsilon \in H^1_0(\Omega) \text{ and } Q\text{-periodic for all } x \in \Omega, \text{ and } f \in L^2(\Omega).$$

Notice that any solution $u^\epsilon$ of (5.1) depends on both $\epsilon$ and $f$:

$$u^\epsilon = (\mathcal{A}^\epsilon)^{-1}f,$$

where

$$\mathcal{A}^\epsilon = -\partial_{x_i}(a^\epsilon(x)\partial_{x_i}).$$

Moreover, observe that $f$ depends on the slow (or macroscopic) variable $x$, whereas $(\mathcal{A}^\epsilon)^{-1}$ depends on the fast (or microscopic) variable $x/\epsilon$. Thus, we see that we may write the solution $u^\epsilon$ of (5.1) as

$$u^\epsilon = u(x,x/\epsilon).$$

Remark 5.1. In our study, the slow variable corresponds to the global structure of the potential field, whereas the fast variable corresponds to the local structure.

To motivate the Fundamental Theorem, we show (formally) that $u^0$ solves the homogenized problem

$$-\partial_{x_i}(a^0_{ij}\partial_{x_j} u^0) = f(x) \text{ in } \Omega,$$

$$u^0 \in H^1_0(\Omega) \text{ and } Q\text{-periodic for all } x \in \Omega, f \in L^2(\Omega), \text{ and }$$

$$a^0_{ij} \text{ are constants for all } i,j = 1,2,\ldots,n.$$
Here, $a_{ij}^0$ is given by the formula

$$a_{ij}^0 = \int_Q a(y)(e_i^j + \partial_y v_j(y)) \, dy,$$

where

$$\partial_y (a(y)(e_i^j + \partial_y v_j(y))) = 0,$$

$$e^j = (0, \ldots, 0, 1, 0, \ldots, 0).$$

**Remark 5.2.** Notice that (5.3) is equivalent to the effective conductivity tensor $A$ defined by (3.5) on page 10, which was derived using purely physical arguments.

Before proceeding to motivate the Fundamental Theorem, we state a result that we shall need.

**Proposition 5.1.** Let $Q \subseteq \mathbb{R}^n$, and let $\alpha$ and $\beta$ be positive real constants. Further, let $a(y)$ be a $Q$-periodic real-valued function such that $0 < \alpha \leq a(y) \leq \beta < \infty$. Then the equation

$$-\partial_y (a(y) \partial_y \phi) = F \text{ in } Q,$$

where $\phi$ is $Q$-periodic, admits a unique solution (up to an additive constant) if and only if

$$\int_Q F(y) \, dy = 0.$$

**Proof.** To see the necessity of (5.5), define the operator $A$ by

$$A = -\partial_y (a(y) \partial_y).$$

Then $A$ is self-adjoint on the class of $Q$-periodic functions in $H^1(\Omega)$, and the nullspace of the adjoint of $A$ is

$$\mathcal{N}(A^*) = \mathcal{N}(A) = \{t: t \text{ constant}\}.$$ 

Thus, condition (5.5) is an instance of the Fredholm Alternative (e.g. [7]).

To see the uniqueness of the solution $\phi$, suppose that $\phi_1$ and $\phi_2$ solve Equation (5.4). Then $\psi = \phi_1 - \phi_2$ satisfies the boundary value problem

$$-\partial_y (a(y) \partial_y \psi) = 0 \text{ in } Q,$$

$$\psi = 0 \text{ on } \partial Q.$$ 

Multiply through the PDE in (5.6) by $-\psi$; then integration by parts and application of the boundary condition give

$$-\int_Q a(y)|\nabla \psi|^2 \, dy = 0.$$
But \( 0 < \alpha \leq a(y) \leq \beta \) implies that \(|\nabla \psi|^2 = 0\) or, equivalently, that \(\nabla \psi = 0\). Therefore, we see that \(\psi\) is a constant. However, \(\psi\) is zero on the boundary of \(Q\), so that it must be zero on the closure of \(Q\); i.e.,

\[
\phi_1 - \phi_2 = \psi = 0 \quad \text{in} \ Q.
\]

\[\square\]

We now proceed to motivate the Fundamental Theorem of Periodic Homogenization.

We begin by assuming that \(u^\varepsilon\) may be expanded asymptotically as

\[
(5.7) \quad u^\varepsilon \approx u^0(x, x/\varepsilon) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon^2 u^2(x, x/\varepsilon) + \cdots,
\]

where, for each \(k = 0, 1, 2, \ldots\), we have \(u^k \in H^1_0(\Omega)\) and \(Q\)-periodic for all \(x \in \Omega\).

Next, let \(y = x/\varepsilon\). Then we have that

\[
\partial_x i u^k(x, x/\varepsilon) = \partial_x i u^k(x, y) = \partial_x i u^k(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = \partial_x i u^k + \partial_y i u^k \cdot \varepsilon^{-1}.
\]

This motivates us to define

\[
(5.8) \quad \tilde{u}^\varepsilon(x, y) \equiv u^0(x, y) + \varepsilon u^1(x, y) + \varepsilon^2 u^2(x, y) + \cdots,
\]

and the operator

\[
\partial_x i \equiv (\partial_x i + \varepsilon^{-1} \partial_y i).
\]

With this operator, we may rewrite (5.1) as

\[
(5.9) \quad -(\partial_x i + \varepsilon^{-1} \partial_y i)(a(y)(\partial_x i + \varepsilon^{-1} \partial_y i)u^\varepsilon) = f(x) \quad \text{in} \ \Omega,
\]

\(u^k \in H^1_0(\Omega)\) and \(Q\)-periodic for all \(x \in \Omega, k = 0, 1, 2, \ldots\), and \(f \in L^2(\Omega)\).

Thus, instead of substituting (5.7) into (5.1), we substitute (5.8) into (5.9) instead, and proceed as if \(x\) and \(y\) were independent variables. This gives us the partial differential equation

\[
-(\partial_x i + \varepsilon^{-1} \partial_y i)(a(y)(\partial_x i + \varepsilon^{-1} \partial_y i)(u^0(x, y) + \varepsilon u^1(x, y) + \cdots)) = f(x),
\]
Using (5.12), we rewrite (5.11) as
\[ -\sum_{k=0}^{\infty} \{ \partial_x (a(y) \partial_x, u^k) \epsilon^k + [\partial_x (a(y) \partial_y, u^k + \partial_y (a(y) \partial_x, u^k)] \epsilon^{k-1} + \partial_y (a(y) \partial_y, u^k) \epsilon^{k-2} \} = f(x). \]

Now, since \( f(x) = 0 \epsilon^{-2} + 0 \epsilon^{-1} + f(x) \epsilon^0 + 0 \epsilon^1 + \cdots \), we equate coefficients of like powers of \( \epsilon \) (for \( k = 0, 1, 2 \)) in (5.10) to obtain the following system of equation.

\[
\begin{align*}
(5.10') \quad & 0 = \partial_y (a(y) \partial_y, u^0) \\
(5.10'') \quad & 0 = \partial_x (a(y) \partial_y, u^0) + \partial_y (a(y) \partial_x, u^0) + \partial_y (a(y) \partial_y, u^1) \\
(5.10''') \quad & f = -\partial_x (a(y) \partial_x, u^0) - \partial_y (a(y) \partial_y, u^1) - \partial_y (a(y) \partial_y, u^2)
\end{align*}
\]

We proceed to solve this system of equations.

First, observe that \( u^0(x, y) = u^0(x) \) solves Equation (5.10'). So, by Proposition 5.1, we conclude that \( u^0(x) \) must be the unique solution (up to an additive constant) of (5.10').

Next, since \( u^0(x, y) = u^0(x) \), we see that the first term on the right-hand side of (5.10'') vanishes. Thus, we may rewrite Equation (5.10'') as
\[ \partial_y (a(y) \partial_x, u^0(x)) + \partial_y (a(y) \partial_y, u^1(x, y)) = 0, \]
or, by linearity of the operator \( \partial_y \), as
\[ \partial_y (a(y) (\partial_x, u^0(x) + \partial_y, u^1(x, y))) = 0. \]

Now we make the ansatz that
\[ u^1(x, y) = v^i(y) \partial_x, u^0(x) + \hat{u}^1(x). \]

Using (5.12), we rewrite (5.11) as
\[ \partial_y (a(y) (\partial_x, u^0(x) + \partial_y, u^1(x, y))) = 0. \]

**Remark 5.3.** Notice that, since we are treating \( x \) and \( y \) as independent variables, it follows from linearity that (5.13) that \( v^i \) solves the “cell problem” that was used in Section 3:
\[ \partial_y (a(y) (\epsilon^i + \partial_y, v^i)) = 0, \]
\[ v^i \quad Q\text{-periodic, and } \quad \int_Q v^i \, dy = 0. \]

Finally, turning to Equation (5.10'''), we rewrite it as
\[ -\partial_y (a(y) \partial_y, u^2(x, y)) = F(x, y), \]
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where

\[ F(x, y) = f(x) + \partial_x(a(y)\partial_x u^0(x)) + \partial_x(a(y)\partial_y u^1(x, y)) + \partial_y(a(y)\partial_x u^1(x, y)). \]

Again, by Proposition 5.1, Equation (5.15) has a unique solution if and only if

\[ \int_Q F(x, y) \, dy = 0; \]

i.e., if and only if

\[
(5.16) \quad \int_Q \left[ f(x) + \partial_x(a(y)\partial_x u^0(x)) \\
+ \partial_x(a(y)\partial_y u^1(x, y)) + \partial_y(a(y)\partial_x u^1(x, y)) \right] \, dy = 0.
\]

Notice that because \(a(y)\) and \(u^1(x, y)\) are \(Q\)-periodic for all \(x \in \mathbb{R}^n\), so is \(\partial_y(a(y)\partial_x u^1(x, y))\); hence, \(\int_Q [\partial_y(a(y)\partial_x u^1(x, y))] \, dy = 0\) and, therefore, (5.16) reduces to

\[
(5.17) \quad -\int_Q \left[ \partial_x(a(y)\partial_x u^0(x)) + \partial_x(a(y)\partial_y u^1(x, y)) \right] \, dy = f(x).
\]

Keeping in mind that we are integrating with respect to \(y\), substituting (5.12) into (5.17) we obtain the homogenized problem

\[-\partial_y\left( \int_Q a(y)\left( e_j^0 + \partial_y v^j(y) \right) \, dy \partial_x u^0 \right) = f(x).\]
SECTION 6

Proof of the Fundamental Theorem of Periodic Homogenization via the energy method of Tartar

Let \( Q = (0,1)^n \), and suppose that \( a(x) \in L^2(Q) \) is a \( Q \)-periodic real-valued function as defined in Section 5. Then, again, extend \( a(x) \) by periodicity to \( \mathbb{R}^n \), and define

\[
a^\varepsilon(x)I = a(x/\varepsilon)I,
\]

where \( I \) is the \( n \times n \) identity matrix.

Let \( \Omega \subseteq \mathbb{R}^n \) be an open bounded set with Lipshitz boundary, and recall that the Fundamental Theorem of Periodic Homogenization states that for all \( f \in H^{-1}(\Omega) \), the solutions \( u^\varepsilon \) of

\[
-\partial_x (a^\varepsilon(x) \partial_x u^\varepsilon) = f(x) \quad \text{in } \Omega,
\]

\[
 u^\varepsilon \in H^1_0(\Omega) \text{ and } Q\text{-periodic for all } x \in \Omega, \text{ and } f \in L^2(\Omega),
\]

converge weakly in \( H^1_0(\Omega) \) (as \( \varepsilon \) tends to zero) to the solution \( u^0 \) of

\[
-\partial_x (a^0_{ij} \partial_x u^0) = f(x) \quad \text{in } \Omega,
\]

\[
 u^0 \in H^1_0(\Omega) \text{ and } f \in L^2(\Omega).
\]

Here, the matrix \( a^0_{ij} \) is the effective tensor defined by (5.3).

To prove the Fundamental Theorem, we show that the matrices \( a^\varepsilon(x)I \) \( G \)-converge to the constant matrix \( a^0_{ij} \) as \( \varepsilon \) tends to zero; i.e., we show that the solutions \( u^\varepsilon \) of (6.1) converge weakly to the solution \( u^0 \) of (6.2) in \( H^1_0(\Omega) \) as \( \varepsilon \) tends to zero.

Our proof uses what is known as the “energy method” due to Luc Tartar [5, 39], and is outlined as follows:

Step 1 Obtain a uniform estimate in the \( H^1 \)-norm on the solutions \( u^\varepsilon \) that is independent of \( \varepsilon \);

Step 2 Deduce the existence of a subsequence \( u^\varepsilon' \) that converges weakly to a limit \( u^0 \) in \( H^1_0(\Omega) \) as \( \varepsilon \) tends to zero;

Step 3 Show by the energy method of Tartar that the limit \( u^0 \) solves the homogenized problem (6.2);
Step 4: Show that, indeed, the whole sequence \( u^\epsilon \) converges weakly to \( u^0 \) in \( H_0^1(\Omega) \) as \( \epsilon \) tends to zero.

Before proceeding with the proof, we state two propositions that are used therein.

**Proposition 6.1 (Lemma 4.1 in [39]).** Let \( \phi \in L^2(Y) \). If we extend it periodically to \( \mathbb{R}^n \), we have

\[
\phi(x/\epsilon) \rightharpoonup -\int_Y \phi(x) \, dx \quad \text{(weak) in } L^2(\Omega)
\]
as \( \epsilon \) tends to zero.

**Proof.** See [39] for the proof. \( \square \)

**Proposition 6.2.** Suppose that \( \eta^\epsilon \in L^2(\Omega) \). Then

\[
\eta^\epsilon p^\epsilon \rightharpoonup \eta^0 p^0 \quad \text{(weak) in } L^2(\Omega) \quad \text{as } \epsilon \to 0
\]
whenever

\[
\eta^\epsilon \to \eta^0 \quad \text{(weak) in } L^2(\Omega) \quad \text{as } \epsilon \to 0, \quad \text{and}
\]

\[
p^\epsilon \to p^0 \quad \text{(strong) in } L^2(\Omega) \quad \text{as } \epsilon \to 0.
\]

**Proof.** It will be enough to show that

\[
\int_Q (\eta^\epsilon p^\epsilon - \eta^0 p^0) \phi \, dx \to 0 \quad \text{as } \epsilon \to 0 \quad \text{for all } \phi \in L^2(\Omega).
\]

To show this, we fix \( \phi \in L^2(\Omega) \). Then we have

\[
\left| \int_\Omega (\eta^\epsilon p^\epsilon - \eta^0 p^0) \phi \, dx \right|
\]

\[
= \left| \int_\Omega (\eta^\epsilon p^\epsilon - \eta^\epsilon p^0 + \eta^\epsilon p^0 - \eta^0 p^0) \phi \, dx \right|
\]

\[
\leq \left| \int_\Omega (p^\epsilon - p^0) \eta^\epsilon \phi \, dx \right| + \left| \int_\Omega (\eta^\epsilon - \eta^0) p^0 \phi \, dx \right| \quad \text{triangle inequality}
\]

\[
\leq ||p^\epsilon - p^0||_{L^2} ||\eta^\epsilon \phi||_{L^2} + \left| \int_\Omega (\eta^\epsilon - \eta^0) p^0 \phi \, dx \right| \quad \text{Schwarz Inequality}
\]

\[
\leq ||p^\epsilon - p^0||_{L^2} \cdot K + \left| \int_\Omega (\eta^\epsilon - \eta^0) p^0 \phi \, dx \right|
\]

where \( ||\eta^\epsilon \phi||_{L^2} \) is bounded uniformly in \( \epsilon \) by a constant \( K \) since \( \eta^\epsilon \) converges to \( \eta^0 \) weakly in \( L^2(\Omega) \) as \( \epsilon \) tends to zero.

From the hypotheses the last two terms converge to zero, thereby proving the Proposition. \( \square \)
We now prove the Fundamental Theorem of Periodic Homogenization following Steps 1–4.

Step 1  The variational formulation (e.g. [17]) of the PDE in (6.1) is
\[ \int_{\Omega} a^\varepsilon(x) \partial_{x_i} u^\varepsilon \, dx = \int_{\Omega} \phi f \, dx \quad \text{for all } \phi \in H^1_0(\Omega). \]
By taking \( \phi = u^\varepsilon \), and denoting \( \partial_{x_i} \) by \( \nabla \), we have
\[ \int_{\Omega} a^\varepsilon(x) |\nabla u^\varepsilon|^2 \, dx = \int_{\Omega} u^\varepsilon f \, dx. \]
Since \( \Omega \) is a bounded open set with Lipshitz boundary, the Poincaré Inequality (e.g. [6]) implies that
\[ \|u^\varepsilon\|_{L^2} \leq K \|\nabla u^\varepsilon\|_{L^2} \]
for some constant \( K > 0 \); therefore, since \( 0 < \alpha \leq a(y) \), we have the inequality
\[ \alpha \|u^\varepsilon\|_{L^2}^2 \leq \alpha K \|\nabla u^\varepsilon\|_{L^2}^2 = K \int_{\Omega} \alpha |\nabla u^\varepsilon|^2 \, dx \leq K \int_{\Omega} a^\varepsilon(x) |\nabla u^\varepsilon|^2 \, dx. \]
Also, by the Schwarz Inequality, we have
\[ \left| \int_{\Omega} u^\varepsilon f \, dx \right| \leq \|u^\varepsilon\|_{L^2} \|f\|_{L^2}. \]
Substituting (6.4) and (6.5) into (6.3) gives the inequality
\[ \alpha \|u^\varepsilon\|_{L^2}^2 \leq \alpha K \|\nabla u^\varepsilon\|_{L^2}^2 \leq K \|u^\varepsilon\|_{L^2} \|f\|_{L^2}, \]
from which it follows that
\[ \|u^\varepsilon\|_{L^2} \leq \alpha^{-1} K \|f\|_{L^2} \leq B_1 \]
for some constant \( B_1 > 0 \) since \( f \in L^2(\Omega) \). Moreover, using this, we also obtain from (6.6) the inequality
\[ \|\nabla u^\varepsilon\|_{L^2} \leq \|\nabla u^\varepsilon\|_{L^2}^2 \leq \alpha^{-1} \|u^\varepsilon\|_{L^2} \|f\|_{L^2} \leq B_2 \]
for some constant \( B_2 > 0 \). Therefore, we obtain the uniform \( H^1 \)-bound on \( u^\varepsilon \) given by
\[ \|u^\varepsilon\|_{H^1} = \|u^\varepsilon\|_{L^2} + \|\nabla u^\varepsilon\|_{L^2} \leq B = B_1 + B_2. \]

Step 2  By the Alaoglu Theorem (e.g. [12]), we see that (6.8) implies that \( \{u^\varepsilon\} \subseteq H^1_0(\Omega) \) is compact in the weak topology. Therefore, there exists a subsequence \( u^\varepsilon \) converging weakly to a limit \( u^0 \) in \( H^1_0(\Omega) \).

Step 3  We now show that the limit \( u^0 \) satisfies the homogenized problem (6.2). To do this, we first observe that the fact that \( a^\varepsilon(x) \in L^2(\Omega) \) together with Inequality (6.7) imply that
\[ \|a^\varepsilon(x) \partial_{x_i} u^\varepsilon\|_{L^2} \leq C \]
for some constant $C > 0$. Therefore, appealing to the Alaoglu Theorem again, we deduce that there exists a subsequence $a^\epsilon' (x) \partial x_i u^\epsilon' \Rightarrow M^0_i$ (weak) in $L^2(\Omega)$ as $\epsilon \to 0$

for some $M^0_i \in L^2(\Omega)$.

**Remark 6.1.** In what follows, we use $\epsilon$ to index all subsequences.

Next, consider the variational equation

$$
\int_{\Omega} a^\epsilon (x) \partial x_i u^\epsilon \partial_x \phi \, dx = \int_{\Omega} \phi f \, dx \quad \text{for all } f \in H^1_0 (\Omega).
$$

Using (6.9), passing to the limit as $\epsilon$ tends to zero in (6.10) yields

$$
\int_{\Omega} M^0_i \partial_x \phi \, dx = \int_{\Omega} \phi f \, dx \quad \text{for all } \phi \in H^1_0 (\Omega);
$$

thus, we see that

$$
-\partial_x M^0_i = f(x) \quad \text{in } \Omega
$$

in the sense of $H^{-1}(\Omega)$ (e.g. [17]).

It is now evident that to identify $u^0$ as the solution to problem (6.2), it is sufficient to show that the limit $M^0_i$ in (6.9) is given by

$$
M^0_i = a^0_{ij} \partial_x_j u^0.
$$

To accomplish this, we use the energy method of Tartar.

We introduce a “corrector function” $\Psi^\epsilon_i (x)$ defined by

$$
\Psi^\epsilon_i (x) = \epsilon (v^i (x/\epsilon) + x_i/\epsilon),
$$

where $v^i$ solves (5.14). From (6.12), it follows that

$$
\Psi^\epsilon_i (x) \to x_i \quad \text{(strong) in } L^2(\Omega) \text{ as } \epsilon \to 0.
$$

Moreover, we see from (6.12) that

$$
\partial_x \Psi^\epsilon_i (x) = \partial_x_i v^i (x/\epsilon) + e_i,
$$

from which we have

$$
\partial_x \Psi^\epsilon_i (x) \to e_i \quad \text{(weak) in } L^2(\Omega) \text{ as } \epsilon \to 0.
$$

Notice that we may rewrite the PDE in (5.3) as

$$
-\partial_y (a(y) \partial_y (v^i (y) + y_i)) = 0;
$$

then rescaling with $y = x/\epsilon$ so that $\partial_y = \epsilon \partial_x$, gives

$$
-\epsilon \partial_x (a^\epsilon (x) \partial_x (\epsilon (v^i (x/\epsilon) + x_i/\epsilon))) = 0.
$$
Thus, from (6.12) and (6.14), we have

\begin{equation}
-\epsilon \partial_{x_i}(a^\epsilon(x)\partial_{x_i} \Psi_{\epsilon}^i(x)) = 0.
\end{equation}

Substituting \( \phi = \psi \Psi_{\epsilon}^i(x) \), where \( \psi \in C_0^\infty(\Omega) \), into (6.10) and applying the product rule to \( \partial_{x_i}(\psi \Psi_{\epsilon}^i(x)) \) we get

\begin{equation}
\int_\Omega a^\epsilon(x) \partial_{x_i} u^\epsilon \Psi_{\epsilon}^i(x) \partial_{x_j} \psi \, dx + \int_\Omega a^\epsilon(x) \partial_{x_i} u^\epsilon \partial_{x_k} \Psi_{\epsilon}^i(x) \partial_{x_k} \psi \, dx = \int_\Omega (\psi \Psi_{\epsilon}^i(x)) f \, dx;
\end{equation}
on the other hand, if we substitute \( \phi = \psi u^\epsilon \), where \( \psi \in C_0^\infty(\Omega) \), into the variational formulation of (6.15) and apply the product rule to \( \partial_{x_i}(\psi u^\epsilon) \) we obtain

\begin{equation}
\int_\Omega a^\epsilon(x) \partial_{x_i} \Psi_{\epsilon}^i(x) u^\epsilon \partial_{x_j} \psi \, dx + \int_\Omega a^\epsilon(x) \partial_{x_i} \Psi_{\epsilon}^i(x) \partial_{x_k} u^\epsilon \partial_{x_k} \psi \, dx = 0.
\end{equation}

Notice that the second term on the left-hand side of (6.16) is precisely the second term on the left-hand side of (6.17); thus, subtracting (6.17) from (6.16) yields

\begin{equation}
\int_\Omega a^\epsilon(x) \partial_{x_i} u^\epsilon \Psi_{\epsilon}^i(x) \partial_{x_j} \psi \, dx - \int_\Omega a^\epsilon(x) \partial_{x_i} \Psi_{\epsilon}^i(x) u^\epsilon \partial_{x_j} \psi \, dx = \int_\Omega (\psi x_i) f \, dx.
\end{equation}

We claim that

\[ \int_\Omega a^\epsilon(x) \partial_{x_i} \Psi_{\epsilon}^i(x) u^\epsilon \partial_{x_j} \psi \, dx \rightarrow \int_\Omega a^0_{ij} u^0 \partial_{x_i} \psi \, dx \quad \text{(weak in } L^2(\Omega)\text{)} \]
as \( \epsilon \) tends to zero. Deferring its proof to the end of Step 3, we use our claim along with (6.9) and (6.13) in passing to the limit as \( \epsilon \) tends to zero in (6.18) to obtain

\begin{equation}
\int_\Omega M^0_{ij} x_i \partial_{x_j} \psi \, dx - \int_\Omega a^0_{ij} u^0 \partial_{x_i} \psi \, dx = \int_\Omega (\psi x_i) f \, dx.
\end{equation}

Next, observe that choosing \( \phi = \psi x_i \), where \( \psi \in C_0^\infty(\Omega) \), in (6.11) yields

\[ \int_\Omega M^0_{ij} \partial_{x_i} (\psi x_j) \, dx = \int_\Omega (\psi x_i) f \, dx \quad \text{as } \epsilon \rightarrow 0. \]

Using this, we integrate the left-hand side of (6.19) by parts; then, rearranging terms we obtain

\begin{equation}
\int_\Omega M^0_{ij} \psi \, dx = \int_\Omega a^0_{ij} \partial_{x_i} u^0 \psi \, dx.
\end{equation}

Since (6.20) holds for all \( \psi \in C_0^\infty(\Omega) \), we have

\[ M^0_{ij} = a^0_{ij} \partial_{x_i} u^0 \]
in the sense of distributions (e.g. \([17]\)) and since \( a^0_{ij} \) is symmetric, we have our desired result,

\[ M^0_{ij} = a^0_{ij} \partial_{x_i} u^0. \]
Finally, it remains for us to prove our claim that
\[ \int_{\Omega} a^\epsilon(x) \partial_x \Psi^\epsilon_i(x) u^\epsilon \cdot \partial_x \psi \, dx \to \int_{\Omega} a^0_{ij} u^0 \cdot \partial_x \psi \, dx \] (weak) in \( L^2(\Omega) \) as \( \epsilon \) tends to zero. To see this, notice that we have
\[ a^\epsilon(x) \partial_x \Psi^\epsilon_i(x) = a^\epsilon(x) \partial_x, \epsilon(v^i(x/\epsilon) + x_i/\epsilon) \] by definition
\[ = a(y)(\partial_{y}, v^j(y) + e^j_i) \] by rescaling with \( y = x/\epsilon \)
\[ = a(y)(\partial_{y}, v^j(y) + e^j_i); \]
thus, because \( a^\epsilon(x) \partial_x, \Psi^\epsilon_i(x) \) is \( \epsilon \)-periodic, Proposition 6.1 and Equation (5.3) imply that
\[ a^\epsilon(x) \partial_x \Psi^\epsilon_i(x) \to \int_Q a(y)(\partial_{y}, v^j + e^j_i) \, dy = a^0_{ij} \] (weak) in \( L^2(\Omega) \).

Also notice that, as a consequence of the Rellich Lemma (e.g. [8]), it follows from Step 2 that
\[ u^\epsilon \to u^0 \] (strong) in \( L^2(\Omega) \) as \( \epsilon \to 0 \).

Therefore, by Proposition 6.2, our claim is proved. \( \square \)

Step 4 What we have done so far is to show that we may extract from the sequence \( u^\epsilon \) of solutions to the Dirichlet problem (6.1) a subsequence \( u^\epsilon \) that converges weakly in \( H^1_0(\Omega) \) to the solution \( u^0 \) of the homogenized problem (6.2) as \( \epsilon \) tends to zero. Thus, to complete the proof of the Fundamental Theorem, it remains for us to show that, indeed, the whole sequence \( u^\epsilon \) converges to \( u^0 \) weakly in \( H^1_0(\Omega) \) as \( \epsilon \) tends to zero. We begin by stating the following definition.

**Definition 6.1.** Let \( X \) be a metric space. Then we say that a sequence \( \{ x_n \} \) in \( X \) does not converge to \( x \) in \( X \) (as \( n \to \infty \)) if for some \( \delta > 0 \) and for all \( N > 0 \) there exists an \( m > N \) such that \( x_m \) is not in the ball \( B(x, \delta) \) centered at \( x \) with radius \( \delta \).

Now, suppose that the whole sequence \( u^\epsilon \) does not converge to \( u^0 \) weakly in \( H^1_0(\Omega) \) as \( \epsilon \) tends to zero. Then, by Definition 6.1, for some \( \delta > 0 \) we have, for each \( N > 0 \), an \( \eta > N \) such that \( u^\epsilon \eta \not\in B(u^0, \delta) \). Since \( u^\epsilon \eta \) is a subsequence of \( u^\epsilon \) it has, by our previous observation, a sub-subsequence that converges weakly to \( u^0 \) in \( H^1_0(\Omega) \) as \( \epsilon \) tends to zero. But this is impossible! Therefore, we conclude that, as \( \epsilon \) tends to zero, the whole sequence \( u^\epsilon \), in fact, converges to \( u^0 \) weakly in \( H^1_0(\Omega) \). This completes Step 4 and the proof of the Fundamental Theorem of Periodic Homogenization.
SECTION 7

Derivation of bounds characterizing the set of effective conductivity tensors of periodic two-phase composite media

Recall from Section 2 that for a mixture of two isotropic materials with conductivities $\alpha$ and $\beta$ ($0 < \alpha < \beta < \infty$) mixed in some specified proportions $\Theta_\alpha$ and $\Theta_\beta = 1 - \Theta_\alpha$, we defined its effective conductivity tensor $A$ by the variational principle

\begin{equation}
\langle A\xi,\xi \rangle = \inf_v \int_Q a(y)|\xi + \nabla v|^2 \, dy,
\end{equation}

where $\xi$ is any vector in $\mathbb{R}^n$, $v$ ranges over all $Q$-periodic functions ($Q \subseteq \mathbb{R}^n$), and $a(y)$ is the $Q$-periodic function defined by

$$a(y) = \alpha \chi_\alpha(y) + \beta \chi_\beta(y),$$

with $\chi_\alpha$ and $\chi_\beta$ the characteristic functions of the $\alpha$- and $\beta$-materials. We also stated in Section 2 that the set of all such effective conductivity tensors of periodic two-phase composites is completely characterized by the following bounds (cf. Theorem 2.1):

\begin{align}
\tag{7.2}
hI &\leq A \leq mI; \\
\tag{7.3}
\sum_{i=1}^n \frac{1}{\lambda_i - \alpha} &\leq \frac{n-1}{m-\alpha} + \frac{1}{h-\alpha}; \\
\tag{7.4}
\sum_{i=1}^n \frac{1}{\beta - \lambda_i} &\leq \frac{n-1}{\beta - m} + \frac{1}{\beta - h};
\end{align}

where

$$h = \left(\int_Q a^{-1}(y) \, dy\right)^{-1} = \left(\frac{\Theta_\alpha}{\alpha} + \frac{\Theta_\beta}{\beta}\right)^{-1},$$

$$m = \int_Q a(y) \, dy = \alpha \Theta_\alpha + \beta \Theta_\beta,$$
and where $I$ is the $n \times n$ identity matrix, and each $\lambda_i$ ($i = 1, 2, \ldots, n$) is an eigenvalue of $A$. We derive these bounds here, beginning with the harmonic mean-arithmetic mean bounds (7.2).

Notice that, because the effective conductivity tensor $A$ is defined by the variational principle (7.1), any choice of $v$ in (7.1) yields an upper bound on $A$:

$$\langle A\xi, \xi \rangle \leq \int_Q a(y)|\xi + \nabla v|^2 \, dy.$$ 

A convenient choice of $v$ is $v \equiv 0$, which yields the arithmetic mean bound.

Establishing the harmonic mean bound is slightly trickier, and requires the use of the Legendre transformation (e.g. [6]) given by

$$(7.5) \quad \langle C\xi, \xi \rangle \geq 2\langle \xi, \sigma \rangle - \langle C^{-1}\sigma, \sigma \rangle$$

for any $n \times n$ positive symmetric matrix $C$, and any vectors $\xi, \sigma \in \mathbb{R}^n$.

Applying the Legendre transform to the right-hand side of (7.1), and integrating by parts gives the inequality

$$(7.6) \quad \langle A\xi, \xi \rangle \geq \inf_v \int_Q \left(2\langle \xi, \sigma \rangle - 2v \text{div} \sigma - a^{-1}(y)|\sigma|^2 \right) \, dy,$$

which holds for all square-integrable vector fields $\sigma \in \mathbb{R}^n$. In order to prevent the right-hand side of (7.6) from approaching $-\infty$ in the infimum over $v$, we restrict $\sigma$ to the class of divergence-free vector fields. Then (7.6) reduces to

$$(7.7) \quad \langle A\xi, \xi \rangle \geq \inf_v \int_Q \left(2\langle \xi, \sigma \rangle - a^{-1}(y)|\sigma|^2 \right) \, dy.$$ 

Finally, maximizing the right-hand side of (7.7) gives the harmonic mean bound on $A$.

The trace bounds (7.3) and (7.4) shall be established using the Hashin-Shtrikman variational principles [14], which we derive presently from (7.1).

We start by adding and subtracting $\gamma|\xi + \nabla v|^2$, where $\gamma$ is a real constant, to the integrand on the right-hand side of (7.1) to get

$$(7.8) \quad \langle A\xi, \xi \rangle = \inf_v \int_Q \left[(a(y) - \gamma)|\xi + \nabla v|^2 + \gamma|\xi + \nabla v|^2 \right] \, dy.$$ 

Suppose $0 < \gamma < \alpha$. Then $a(y) - \gamma > 0$ so that, by the Legendre transform (7.5), we have

$$(a(y) - \gamma)|\xi + \nabla v|^2 \geq 2\langle \xi + \nabla v, \sigma \rangle - (a(y) - \gamma)^{-1}|\sigma|^2,$$
where \( \sigma \) is a square-integrable vector field. Substituting this into (7.8), and integrating by parts gives

\[
(7.9) \quad \langle A \xi, \xi \rangle \geq \int_Q \left[ 2 \langle \xi, \sigma \rangle - (a(y) - \gamma)^{-1} |\sigma|^2 + \gamma |\xi|^2 \right] + \inf_v \int_Q [\gamma |\nabla v|^2 - 2v \operatorname{div} \sigma] \, dy.
\]

For fixed \( \sigma \), the best choice of \( v \) solves

\[
(7.10) \quad -\gamma \operatorname{div} \nabla v^* = \operatorname{div} \sigma.
\]

Substitute the minimizer \( v^* \) for \( v \) in (7.9), and integrate the second integral by parts; then rearrange terms to obtain the Hashin-Shtrikman variational principle for bounding \( A \) from below:

\[
(7.11) \quad \langle (A - \gamma I) \xi, \xi \rangle \geq \int_Q \left[ 2 \langle \xi, \sigma \rangle - (a(y) - \gamma)^{-1} |\sigma|^2 + \langle \nabla v^*, \sigma \rangle \right] \, dy,
\]

where \( I \) is the \( n \times n \) identity matrix.

The Hashin-Shtrikman variational principle for bounding \( A \) from above, which is given by

\[
(7.12) \quad \langle (A - \gamma I) \xi, \xi \rangle \leq \int_Q \left[ 2 \langle \xi, \sigma \rangle - (a(y) - \gamma)^{-1} |\sigma|^2 + \langle \nabla v^*, \sigma \rangle \right] \, dy,
\]

can be similarly obtained from (7.8) by letting \( \beta < \gamma < \infty \) so that \( a(y) - \gamma < 0 \).

We shall use the following two propositions to obtain the lower trace bound (7.3) from the Hashin-Shtrikman variational principle (7.11).

**Proposition 7.1.** Suppose \( v^* \) and \( \sigma \) satisfy (7.11), and denote the Fourier coefficients of \( \sigma \) by \( \hat{\sigma}(k) \), where \( k \in \mathbb{Z}^n \). Then

\[
\int_Q \langle \nabla v^*, \sigma \rangle \, dy = -\frac{1}{\gamma} \sum_{k \neq 0} \left( \frac{\langle ik, \hat{\sigma}(k) \rangle}{|k|} \right)^2.
\]

**Proof.** Let \( \hat{v}^*(k) \) denote the Fourier coefficients of \( v^* \). Then we have

\[
(7.13) \quad \operatorname{div} \nabla v^* = \operatorname{div} \nabla \left( \sum_k \langle \hat{v}^*(k) \cdot \exp(i \langle k, x \rangle) \rangle \right) = \sum_k ( -|k|^2 \hat{v}^*(k) \cdot \exp(i \langle k, x \rangle) ),
\]

and

\[
(7.14) \quad \operatorname{div} \sigma = \operatorname{div} \sum_k ( \hat{\sigma}(k) \cdot \exp(i \langle k, x \rangle) ) = \sum_k ( \langle ik, \hat{\sigma}(k) \rangle \exp(i \langle k, x \rangle) )).
\]

Substitute (7.13) and (7.14) into the differential equation (7.10), equate coefficients, and solve for \( \hat{v}^*(k) \) to obtain

\[
\hat{v}^*(k) = \frac{\langle ik, \hat{\sigma}(k) \rangle}{\gamma |k|^2}.
\]
Substituting this into the Fourier expansion of $\nabla v^*$ and evaluating then gives
\begin{equation}
\hat{v}^*(k) = -\frac{1}{\gamma} \left( \frac{\langle k, \hat{\sigma}(k) \rangle}{|k|^2} \right) k.
\end{equation}

Finally, our desired result follows from (7.15) and the Plancherel Theorem (e.g. [38]):
\begin{align*}
\int_Q \langle \nabla v^*, \sigma \rangle \, dy &= \sum_{k \neq 0} \langle \nabla v^*(k), \hat{\sigma}(k) \rangle = -\frac{1}{\gamma} \sum_{k \neq 0} \left( \frac{\langle k, \hat{\sigma}(k) \rangle}{|k|} \right)^2.
\end{align*}

\begin{proposition}
For $k \in \mathbb{Z}^n$, let $\hat{\chi}_\beta(k)$ denote the Fourier coefficients of the characteristic function of the $\beta$-material, and define the matrix $F$ by
\begin{equation}
F_{ij} = \sum_{k \neq 0} \left( |\hat{\chi}_\beta(k)|^2 \frac{k_i k_j}{|k|^2} \right).
\end{equation}
Then the trace of $F$ is given by
\begin{equation}
\text{tr}(F) = \Theta_\beta(1 - \Theta_\beta),
\end{equation}
where $\Theta_\beta$ is the volume fraction of the $\beta$-material.
\end{proposition}

\begin{proof}
Observe that
\begin{equation}
\text{tr}(F) = \text{tr} \left[ \sum_{k \neq 0} \left( |\hat{\chi}_\beta(k)|^2 \frac{k_i k_j}{|k|^2} \right) \right] = \sum_{k \neq 0} |\hat{\chi}_\beta(k)|^2.
\end{equation}
The Proposition then follows from the Plancherel Theorem:
\begin{equation}
\sum_{k \neq 0} |\hat{\chi}_\beta(k)|^2 = \int_Q (\chi_\beta(y) - \Theta_\beta)^2 \, dy = \Theta_\beta(1 - \Theta_\beta).
\end{equation}
\end{proof}

We now derive the lower bound (7.3).

By Proposition 7.1, we may rewrite the Hashin-Shtrikman variational principle (7.11) as
\begin{equation}
\langle (A - \gamma I) \xi, \xi \rangle \geq \int_Q \left[ 2 \langle \xi, \sigma \rangle - (a(y) - \gamma)^{-1} |\sigma|^2 \right] \, dy - \frac{1}{\gamma} \sum_{k \neq 0} \left( \frac{\langle k, \hat{\sigma}(k) \rangle}{|k|} \right)^2,
\end{equation}
which holds for all vectors $\xi \in \mathbb{R}^n$, and for all square-integrable $Q$-periodic vector fields $\sigma \in \mathbb{R}^n$. Choosing our test field to be $\chi_\beta(y)\eta$, where $\eta$ is a constant vector in $\mathbb{R}^n$, and letting $\gamma$ tend to $\alpha$ in (7.16) gives
\begin{equation}
\langle (A - \alpha I) \xi, \xi \rangle \geq 2\Theta_\beta \langle \xi, \eta \rangle - (\beta - \alpha)^{-1} \Theta_\beta |\eta|^2 - \frac{1}{\alpha} \sum_{k \neq 0} \left( |\hat{\chi}_\beta(k)|^2 \left( \frac{\langle k, \eta \rangle}{|k|} \right)^2 \right).
\end{equation}
Thus, we see that the best lower bound on $A$ can be obtained by fixing $\xi$, and maximizing the right-hand side of (7.17) over all $\eta$. 
To do this, we rewrite (7.17) as

\begin{equation}
\langle (A - \alpha I)\xi, \xi \rangle \geq 2\Theta_\beta \langle \xi, \eta \rangle - \langle L\eta, \eta \rangle,
\end{equation}

where

\[ L = \left( (\beta - \alpha)^{-1} \Theta_\beta I - \frac{1}{\alpha} F \right), \]

\[ F_{ij} = \sum_{k \neq 0} \left( |\hat{\chi}_\beta(x)|^2 \frac{k_ik_j}{|k|^2} \eta_i \eta_j \right) = \sum_{k \neq 0} \left( |\hat{\chi}_\beta(k)|^2 \left( \frac{\langle k, \eta \rangle}{|k|} \right)^2 \right). \]

Then the best choice of \( \eta \) is

\[ \eta^* = \Theta_\beta L^{-1} \xi. \]

Substituting the maximizer \( \eta^* \) into (7.18) yields

\[ \langle (A - \alpha I)\xi, \xi \rangle \geq \Theta_\beta^2 L^{-1}|\xi|^2. \]

Since this holds for all \( \xi \in \mathbb{R}^n \), it follows that

\begin{equation}
A - \alpha I \geq \Theta_\beta^2 L^{-1}.
\end{equation}

Appealing to [4], we invert (7.19); then using Proposition 7.2 in taking the trace of both sides gives

\begin{equation}
\text{tr}\left[(A - \alpha I)^{-1}\right] \leq \Theta_\beta^2 \left[ \frac{n\Theta_\beta}{\beta - \alpha} + \frac{\Theta_\beta(1 - \Theta_\beta)}{\alpha} \right].
\end{equation}

A straightforward (albeit lengthy) calculation shows that (7.20) is, indeed, the lower trace bound (7.3).

The upper trace bound (7.4) can be similarly derived from the Hashin-Shtrikman variational principle (7.12).
SECTION 8

Attainability of bounds characterizing the set of effective conductivity tensors of periodic two-phase composites

We recall the statement of Theorem 2.1: For any symmetric positive definite matrix \( A \) satisfying (7.2)–(7.4), one can construct a composite material with effective conductivity tensor given by \( A \) using two isotropic materials with conductivities \( \alpha \) and \( \beta \) \((0 < \alpha < \beta < \infty)\) mixed in some specified proportions \( \Theta_\alpha \) and \( \Theta_\beta = 1 - \Theta_\alpha \).

We shall prove Theorem 2.1 in two parts: first, we show that the bounds (7.3) and (7.4) are attainable subject to (7.2) (Lemma 8.1), and then we show that the eigenvalues lying inside these bounds are also attainable (Lemma 8.2). Our composite achieving these bounds, and all points in between, will be constructed using the well-known method of successive lamination.

The method of successive lamination is an iterative construction producing microstructures having several different length-scales. A laminar composite of rank 1 is obtained by layering two materials \( \alpha_1 \) and \( \beta_1 \) in some specified proportions and layer direction, and with some length-scale \( \epsilon_1 \) (representing the layer thickness). As \( \epsilon_1 \) tends to zero, the effective conductivity of the laminar composite is given by an effective tensor \( A_1 \). A laminar composite of rank 2 is obtained by layering two rank-1 laminar composites, again in some specified proportion and layer direction, and with some length-scale \( \epsilon_2 = o(\epsilon_1) \). As \( \epsilon_1 \) and \( \epsilon_2 \) tend to zero, the effective conductivity of the composite is given by an effective tensor \( A_2 \). And, by repeating this procedure, we can construct a laminar composite of any finite rank. Such composites have been discussed by many authors, for example \([1, 2, 9, 10, 11, 20, 23] - [31], [40, 45] \), and have been used to prove the attainability of many different bounds.

The cornerstone for such a successive-laminate construction here is a general formula for the effective conductivity tensor of a composite formed by layering two (possibly anisotropic) media in some specified volume fractions and layer direction. This general formula is stated in (8.1) and (8.2) in the following proposition.
Proposition 8.1 (Master Formula in [18], p. 10; also see [45]). Consider a composite material constructed by laminating two (possibly anisotropic) media with conductivity tensors given by $A$ and $B$ in the proportions $\Theta$ and $1 - \Theta$, respectively, and layered in the direction $\omega \in \mathbb{R}^n$. Then the effective conductivity tensor $A$ of the composite is given by

\begin{equation}
(A - A)^{-1} = \frac{1}{1 - \Theta} (B - A)^{-1} + \frac{\Theta}{1 - \Theta} \omega \otimes \omega, \tag{8.1}
\end{equation}

and

\begin{equation}
(A - B)^{-1} = \frac{1}{\Theta} (A - B)^{-1} + \frac{1 - \Theta}{\Theta} \omega \otimes \omega, \tag{8.2}
\end{equation}

where, for

$$
\omega = \begin{pmatrix}
\omega_1 \\
\omega_2 \\
\vdots \\
\omega_n
\end{pmatrix},
$$

$$
\omega \otimes \omega = \begin{pmatrix}
\omega_1 \omega_1 & \omega_1 \omega_2 & \cdots & \omega_1 \omega_n \\
\omega_2 \omega_1 & \omega_2 \omega_2 & \cdots & \omega_2 \omega_n \\
\vdots & \vdots & \ddots & \vdots \\
\omega_n \omega_1 & \omega_n \omega_2 & \cdots & \omega_n \omega_n
\end{pmatrix}.
$$

Remark 8.1. $\omega \otimes \omega$ is the projection operator. To see this, observe that for any $\eta \in \mathbb{R}^n$

$$
\frac{\omega \otimes \omega}{|\omega|^2} \eta = \frac{\langle \eta, \omega \rangle}{|\omega|^2} \omega.
$$

Proof of Proposition 8.1. Formula (8.1) is obtained by solving the following system of linear equations. Here $\xi$, $\xi_A$, $\xi_B$, and $t$ are vectors in $\mathbb{R}^n$ with $\langle t, \omega \rangle = 0$.

\begin{align*}
\langle \xi_A, t \rangle &= \langle \xi_B, t \rangle \tag{8.3a} \\
\langle A \xi_A, \omega \rangle &= \langle B \xi_B, \omega \rangle \tag{8.3b} \\
A \xi &= \Theta A \xi_A + (1 - \Theta) B \xi_B \tag{8.3c} \\
\xi &= \Theta \xi_A + (1 - \Theta) \xi_B \tag{8.3d}
\end{align*}

Postponing the derivation of the system (8.3a)–(8.3b), we first solve (8.1).

Solving for $\Theta \xi_A$ in (8.3d) gives

$$
\Theta \xi_A = \xi - (1 - \Theta) \xi_B;
$$
substituting this into (8.3c) yields

\[(8.4) \quad (A - A)\xi = (1 - \Theta)(B - A)\xi_B.\]

Let \( \mu = (A - A)\xi \). It then follows from (8.4) that

\[(8.5) \quad \xi_B = \frac{(B - A)^{-1}}{1 - \Theta} \mu.\]

Next, from (8.3a) we have

\[(8.6) \quad \xi_A = \xi_B + \lambda\omega\]

for some real constant \( \lambda \). Using this in (8.3b), a straightforward calculation gives

\[(8.7) \quad \lambda = \frac{\langle \omega, \mu \rangle}{(1 - \Theta)\langle A\omega, \omega \rangle}.\]

Finally, substitution yields

\[(A - A)^{-1}\mu = \xi \]

\[= \Theta\xi_A + (1 - \Theta)\xi_B \quad \text{by (8.3d)}\]
\[= \Theta(\xi_B + \lambda\omega) + (1 - \Theta)\xi_B \quad \text{by (8.6)}\]
\[= \xi_B + \Theta\lambda\omega\]
\[= \frac{(B - A)^{-1}}{1 - \Theta} \mu + \Theta \left[ \frac{1}{1 - \Theta} \langle A\omega, \omega \rangle \right] \omega \quad \text{by (8.5) and (8.7)}\]
\[= \frac{(B - A)^{-1}}{1 - \Theta} \mu + \frac{\Theta}{1 - \Theta} \omega \otimes \omega \mu \quad \text{by Remark 8.1,}\]

from which Formula (8.1) follows.

It remains for us to derive the system of equations (8.3a)--(8.3d).

Recall that the effective conductivity tensor \( A \) of a \( Q \)-periodic medium \((Q \subseteq \mathbb{R}^n)\) composed of two materials with effective conductivity tensors \( A \) and \( B \) mixed in the volume fractions \( \Theta \) and \( 1 - \Theta \), respectively, is given by

\[(8.8) \quad \langle A\xi, \xi \rangle = \inf_v \int_Q a(y)(\xi + \nabla v)^2 \, dy,\]

where \( \xi \) is any vector in \( \mathbb{R}^n \), \( v \) ranges over all \( Q \)-periodic functions, and

\[a(y) = \xi_A(y) A + \chi_B(y) B,\]

where \( \chi_A \) and \( \chi_B \) are the characteristic functions of the \( A \)- and \( B \)-materials. Since the minimizer \( v^\xi \) of (8.8) solves the cell problem, we have

\[(8.9) \quad \text{div}(a(y)(\xi + \nabla v^\xi)) = 0, \quad \text{and}\]
\[(8.10) \quad \left[ a(y)(\xi + \nabla v^\xi) \right]^B_A = 0,\]
with \( v = \langle \xi, x \rangle + v^\xi \) continuous, and \( \xi + v^\xi \) \( Q \)-periodic. Thus, it follows from (8.8) that

\begin{equation}
A\xi = \int_Q a(y)|\xi + \nabla v^\xi|dy.
\end{equation}

To derive the system (8.3a)–(8.3d), we suppose that a period cell of our layered composite is given (schematically) by Figure 8.1(a), where \( \omega \) is the layer direction, and \( t \) is the unit vector tangent to the phase boundary (so that \( \langle t, \omega \rangle = 0 \)). For this geometry, we assume that the solution \( v \) is piecewise linear and continuous in \( Q \); i.e., we assume that the solution is of the form

\[ h(x) = \langle \xi, x \rangle + v^\xi, \]

as depicted (schematically) in Figure 8.1(b). Then

\begin{equation}
h(x) = \xi + \nabla v^\xi = \begin{cases} \xi_A & \text{in the } A\text{-material} \\ \xi_B & \text{in the } B\text{-material} \end{cases},
\end{equation}

where \( \xi_A \) and \( \xi_B \) are vectors in \( \mathbb{R}^n \). Also, since \( h(x) \) is continuous across the phase boundary, we have the consistency condition

\begin{equation}
\langle \nabla h, t \rangle_A = \langle \nabla h, t \rangle_B.
\end{equation}

At this point, we note that Equation (8.3a) follows from (8.12) and (8.13), and that Equation (8.3b) follows from (8.10) and (8.12).
Next, since \( \int_Q \chi_A(y) \, dy = \Theta \) and \( \int_Q \chi_B(y) \, dy = 1 - \Theta \), (8.11) and (8.12) yield
\[
A \xi = - \int_Q a(y) |\xi + \nabla v^\xi| \, dy
= \int_Q (\chi_A(y) A \xi_A + \chi_B(y) B \xi_B) \, dy
= \Theta A \xi_A + (1 - \Theta) B \xi_B,
\]
which is precisely Equation (8.3c).

Lastly, since \( v^\xi \) is \( Q \)-periodic, using (8.12) we have
\[
\xi = - \int_Q (\xi + \nabla V^\xi) \, dy = \Theta A \xi_A + (1 - \Theta) B \xi_B,
\]
which is literally Equation (8.3d).

Thus, we have derived the system of linear equations (8.3a)--(8.3d), thereby completing the proof of Formula (8.1).

Formula (8.2) is proved similarly by interchanging the roles of \( A \) and \( B \), and of \( \Theta \) and \( 1 - \Theta \); i.e., by renaming
\[
A \leftrightarrow B \quad \text{and} \quad \Theta \leftrightarrow 1 - \Theta.
\]
\[\Box\]

Now consider a sequence \( A_1, A_2, \ldots \) of effective conductivity tensors such that \( A_1 \) is the effective tensor of a composite made by successively laminating two isotropic materials with conductivities \( \alpha \) and \( \beta \) \((0 < \alpha < \beta < \infty)\) in the proportions \( \rho_1 \) and \( 1 - \rho_1 \), respectively, using the layer direction \( \omega_1 \); \( A_k \) \((k \geq 2)\) is the effective tensor of a medium made by layering the composite \( A_{k-1} \) into the \( \alpha \)-material in the proportions \( \rho_k \) of the \( \alpha \)-material and \( 1 - \rho_k \) of the \( A_{k-1} \)-material, using the layer direction \( \omega^k \).

Evidently, \( A_k \) represents the effective behavior of a certain laminar composite of rank \( k \), with total volume fraction of the \( \alpha \)-material given by
\[
\zeta_k = 1 - \prod_{i=1}^{k} (1 - \rho_i).
\]
By iterating (8.1), we obtain a formula for \( A_k \):
\[
(8.14) \quad (A_k - \alpha I)^{-1} = \frac{(\beta - \alpha)^{-1}}{1 - \zeta_k} I + \frac{1}{\alpha(1 - \zeta_k)} \sum_{i=1}^{k} \{(\zeta_i - \zeta_{i-1}) \omega^i \otimes \omega^i\},
\]
where \( \zeta_0 = 0 \), and \( I \) is the \( n \times n \) identity matrix. We use (8.14) in the following proposition.
Proposition 8.2. Fix an integer \( n \geq 1 \); standard basis vectors \( e_1, e_2, \ldots, e_n \in \mathbb{R}^n \); real numbers \( w_1, w_2, \ldots, w_n \), with \( 0 \leq w_1, w_2, \ldots, w_n \leq 1 \) and \( \sum_{i=1}^{n} w_i = 1 \); and real number \( \Theta_\alpha \) with \( 0 < \Theta_\alpha < 1 \). Then there is a laminar composite of rank \( n \) made by successively laminating two isotropic materials (as described above) with conductivities \( \alpha \) and \( \beta \), \( 0 < \alpha < \beta < \infty \), in the proportions \( \Theta_\alpha \) and \( \Theta_\beta = 1 - \Theta_\alpha \), and whose effective conductivity tensor \( A \) is given by

\[
(A - \alpha I)^{-1} = \left( \frac{\beta - \alpha}{\Theta_\beta} \right) I + \frac{\Theta_\alpha}{\alpha \Theta_\beta} \sum_{i=1}^{n} \{ w_i e_i \otimes e_i \},
\]

where \( I \) is the \( n \times n \) identity matrix.

Proof. Choose \( k = n \), \( \omega^i = e^i \) \((i = 1, 2, \ldots, n)\), and set \( \zeta_i - \zeta_{i-1} = \Theta_\alpha w_i \) in (8.14). \( \square \)

Lemma 8.1. The lower trace bound (7.3) and the upper trace bound (7.4) are attainable subject to (7.2); i.e., for each of these bounds, one can construct a laminar composite with effective conductivity tensor \( A \) whose eigenvalues lie on the respective trace bounds subject to (7.2).

Remark 8.2. In the 2-dimensional case, the intersection of the trace bounds (7.3)–(7.4) lies entirely within the harmonic mean-arithmetic mean bounds (7.2) (see Figure 8.2); this is not so in dimensions \( n > 2 \). To ease the prose, however, we shall simply refer to the attainability of the trace bounds (7.3)–(7.4), with the fact that the trace bounds are attainable subject to (7.2) being understood.

Remark 8.3. We prove the attainability of the lower bound (7.3); the proof of the attainability of the upper bound (7.4) is analogous, and is accomplished by constructing a rank-\( n \) laminar composite made by layering into the \( \beta \)-material vice the \( \alpha \)-material.
Proof of Lemma 8.1. Taking the trace of both sides of (8.15) yields

\[ \text{tr}[(A - \alpha I)^{-1}] = \frac{n}{(\beta - \alpha)\Theta_\beta} + \frac{\Theta_\alpha}{\alpha\Theta_\beta}, \]

which is equivalent to (7.20), that is, the lower bound (7.3). Thus, we see that the eigenvalues of the effective conductivity tensor \( A \) of a laminar composite constructed by layering into the \( \alpha \)-material as described above lie on (7.3). Therefore, to prove the attainability of the lower bound (7.3), it will be enough for us to show that we may obtain every point on (7.3) by varying the parameters \( \rho_i \) (\( i = 1, 2, \ldots, n \)) in (8.15), where the \( \rho_i \) are given in terms of the weights \( w_i \). To fix ideas, we turn to the 2-dimensional case (see Figure 8.2).

Let \( n = 2 \). Then (8.15) may be written as

\[
A = \begin{pmatrix}
\alpha + \frac{\alpha(\beta - \alpha)\Theta_\beta}{\alpha + (\beta - \alpha)\rho_1} & 0 \\
0 & \alpha + \frac{\alpha(\beta - \alpha)\Theta_\beta}{\alpha + (\beta - \alpha)(\Theta_\alpha - \rho_1)}
\end{pmatrix}.
\]

A straightforward calculation shows that the eigenvalue

(8.16)
\[
\alpha + \frac{\alpha(\beta - \alpha)\Theta_\beta}{\alpha + (\beta - \alpha)\rho_1}
\]

of \( A \) is equivalent to the harmonic mean bound and the arithmetic mean bound (cf. (7.2)) when \( \rho_1 \) is \( \Theta_\alpha \) and 0, respectively; on the other hand, the eigenvalue

(8.17)
\[
\alpha + \frac{\alpha(\beta - \alpha)\Theta_\beta}{\alpha + (\beta - \alpha)(\Theta_\alpha - \rho_1)}
\]

is equivalent to the harmonic mean bound and the arithmetic mean bound when \( \rho_1 \) is 0 and \( \Theta_\alpha \), respectively. Moreover, as a function of \( \rho_1 \), the eigenvalue (8.16) is monotone decreasing, whereas the eigenvalue (8.17) is monotone increasing. Therefore, we conclude that the eigenvalues (8.16) and (8.17) of \( A \) sweep out the entire lower bound (7.3) as \( \rho_1 \) ranges from 0 to \( \Theta_\alpha \).

This proves the attainability of the lower bound (7.3) subject to (7.2) in the 2-dimensional case; the \( n \)-dimensional case is handled similarly.

Thus, we have completed the first part of the proof of Theorem 2.1; to finish the proof of the Theorem, it remains for us to show that we can construct a composite with effective conductivity tensor whose eigenvalues lie inside the bounds (7.2)–(7.4). To fix ideas, we again consider the 2-dimensional case; the \( n \)-dimensional case is handled analogously.

Lemma 8.2. Suppose that \( (\lambda_L, \lambda_2) \) lies on the lower bound (7.3), and that \( (\lambda_U, \lambda_2) \) lies on the upper bound (7.4). Then for each pair of eigenvalues \( (\lambda, \lambda_2) \) with
\[ \lambda_L \leq \lambda \leq \lambda_U, \text{ one can construct a composite material with effective conductivity tensor } A \text{ given by} \]
\[ A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda_2 \end{pmatrix}. \]

(See Figure 8.3.)

**Proof of Lemma 8.2.** Note that it is sufficient for us to construct a composite with effective conductivity tensor \( A \) given by
\[ A = \begin{pmatrix} \Theta \lambda_L + (1 - \Theta) \lambda_U & 0 \\ 0 & \lambda_2 \end{pmatrix}, \]
where \( 0 \leq \Theta \leq 1 \).

Consider two composite media with effective tensors \( A_L \) and \( A_U \) given by
\[ A_L = \begin{pmatrix} \lambda_L & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \text{and} \quad A_U = \begin{pmatrix} \lambda_U & 0 \\ 0 & \lambda_2 \end{pmatrix}. \]

Our constructed medium will be a laminar composite of rank 3 made by layering \( A_L \) and \( A_U \) in the proportions \( \Theta \) and \( 1 - \Theta \), respectively, and in the direction \( e^2 = (0,1) \).

For \( Q \subseteq \mathbb{R}^2 \), define the \( Q \)-periodic function \( a(y) \) by
\[ a(y) = \chi(y) A_L + (1 - \chi(y)) A_U, \]
where \( \chi \) is the characteristic function of the \( A_L \)-material, so that \( \int_Q \chi(y) \, dy = \Theta \).
Then the effective conductivity tensor of our proposed medium is defined by the relation
\[ A \xi = \int_Q a(y) |\xi + \nabla v| \, dy, \]
where \( \xi \) is any vector in \( \mathbb{R}^2 \), and \( v \) is a \( Q \)-periodic function that satisfies

\[
\text{div}(a(y)(\xi + \nabla v)) = 0, \tag{8.22}
\]

\[
\left[ a(y)(\xi + \nabla v) \right]_{AL}^A = 0. \tag{8.23}
\]

Indeed, since our proposed composite is layered in the direction \( e^2 \), we see that \( a(y) \) in (8.20) may be written as

\[
a(y \cdot e^2) = a(y_2) = \begin{pmatrix}
\lambda_L \chi(y_2) + \lambda_U (1 - \chi(y_2)) & 0 \\
0 & \lambda_2
\end{pmatrix}.
\]

To prove that \( A \) is given by (8.19), we show that \( \xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are eigenvectors of \( A \) with corresponding eigenvalues \( \mu_1 \) and \( \mu_2 \) given by

\[
\mu_1 = \Theta \lambda_L + (1 - \Theta) \lambda_U, \quad \text{and} \quad \mu_2 = \lambda_2.
\]

To show that \( \xi_1 \) is an eigenvector of \( A \) with eigenvalue \( \mu_1 \), substitute \( \xi_1 \) into (8.22) to obtain the PDE

\[
\partial_{y_1} (a_{11}(y_2) (1 + \partial_{y_1} v^1)) + \partial_{y_2} (a_{22}(y_2) \partial_{y_2} v^1) = 0. \tag{8.24}
\]

Since \( v^1 \equiv \text{constant} \) satisfies (8.24) and the jump condition (8.23), we conclude by (8.21) that

\[
A\xi_1 = \int_Q a(y_2) \left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \partial_{y_1} v^1 \\ \partial_{y_2} v^1 \end{pmatrix} \right] dy
= \begin{pmatrix} \int_Q a(y_2) dy \\ 0 \end{pmatrix} \quad v^1 \equiv \text{constant}
= \mu_1 \xi_1.
\]

To show that \( \xi_2 \) is an eigenvector of \( A \) with eigenvalue \( \mu_2 \), substitute \( \xi_2 \) into (8.21) to obtain the PDE

\[
\partial_{y_1} (a_{11}(y_2) \partial_{y_1} v^2) + \partial_{y_2} (a_{22}(y_2) (1 + \partial_{y_2} v^2)) = 0. \tag{8.25}
\]

Notice that \( v^2 = v^2(y_2) \) satisfies the jump condition (8.23), and causes the first term on the left-hand side of (8.25) to vanish. Thus, substituting \( v^2 = v^2(y_2) \) into (8.25) yields

\[
a_{22}(y_2) (1 + \partial_{y_2} v^2) = C, \tag{8.26}
\]

for some constant \( C \). Since \( a_{22}(y_2) \geq \lambda_L > 0 \), we may divide through (8.26) by \( a_{22}(y_2) \); doing so, integration gives

\[
C = \lambda_2.
\]
Therefore, we conclude by (8.21) that
\[ A\xi^2 = \mu_2\xi^2. \]

This completes the proof of Theorem 2.1.

Remark 8.4. If, in constructing the composite given by (8.18), we had layered the media \( A_L \) and \( A_U \) in the direction \( e^1 = (\frac{1}{0}) \), we would have obtained the effective conductivity tensor \( A \) given by
\[
A = \begin{pmatrix}
\left( \frac{\mu}{\lambda_L} + \frac{1-\mu}{\lambda_U} \right)^{-1} & 0 \\
0 & \lambda_2
\end{pmatrix},
\]
vice that given by (8.19).
Bibliography


