Some Derivatives of Newton’s Method

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Abstract
Every first-year calculus student learns Newton’s method as part of a repertoire of numerical root-finding methods. We present two other numerical root-finding methods that we derive using Newton’s method: a simplified Newton’s method, and Halley’s method. These two methods are not new, yet they seldom find their way into a first calculus course even though they would not be out of place in such a course.

KEYWORDS: Newton’s method, Halley’s method, numerical root-finding, dynamical systems, fixed point, orbit, iteration function.

1 Introduction
Every first-year calculus student learns Newton’s method (N.M.) as part of a repertoire of numerical root-finding methods; see [12], for example. Of the variety of root-finding methods introduced, N.M. is a particularly important one for several reasons: it is a marvelous application of the differential calculus; it is

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widely applicable; it is relatively fast, being quadratically convergent,\(^1\) while not being computationally too expensive. And it even leads to beautiful pictures that have found their way into art galleries, coffee-table books, and everyday homes [7].

We present two other numerical root-finding methods that we derive using N.M. (hence, the title, “Some derivatives of Newton’s method”): a simplified Newton’s method (S.N.M.), and Halley’s method (H.M.). These two methods are not new, yet they seldom find their way into a first calculus course even though they would not be out of place in such a course. We hope that this note will encourage the classroom presentation of these other two methods.

This note is organized as follows. In Section 2, we introduce the methods and give a geometrical interpretation for each method. In Section 3, we compare the methods by using each of them to approximate \(\sqrt{2}\). In Section 4, we take a dynamical systems approach to the methods, and use this point of view to make a second comparison of the methods, as well as to discuss briefly a couple of ways in which the methods may fail.

2 The methods

The basic problem is to find a root, \(\alpha\), of a given equation,

\[
f(x) = 0,
\]

without having to solve the equation. At the heart of the three numerical root-finding methods we discuss is the following: make an initial guess, \(x_0\), for \(\alpha\), and then use a recurrence formula to obtain a sequence, \(x_0, x_1, x_2, \ldots\), that converges to \(\alpha\). We assume throughout that \(\alpha\) is a simple root of (1), that is, \(f'(\alpha) \neq 0\).

2.1 Newton’s method (N.M.)

“Go west, young man!” was the advice given many of our nation’s early settlers. When it comes to solving equations, an appropriate cry may be, “Linearize, young student!” And linearization is, indeed, the basis for Newton’s method\(^1\)

\(\text{This means that an answer that is correct to one decimal place at one step should be correct to two decimal places at the next step, four decimal places at the following step, and so on} [6, p. 40].\)
Figure 1: Newton’s method.

\[(N.M.): \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{2}\]

To derive the recurrence formula (2), we first linearize (1) about \(x = x_0\), that is to say, we replace \(f\) in (1) by its 1st-degree Taylor polynomial centered at \(x = x_0\). This gives

\[f(x_0) + f'(x_0)(x - x_0) = 0. \tag{3}\]

The linear equation (3) can be solved easily, and we suppose that \(x_1\) is the solution:

\[x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.\]

Geometrically, \(x_1\) is the \(x\)-intercept of the line that is tangent to the graph of \(y = f(x)\) at the point \((x_0, f(x_0))\).

We repeat the process by linearizing (1) about \(x = x_1\) next. Then \(x_2\) is the solution of the new linearized equation,

\[x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},\]

if and only if \(x_2\) is the \(x\)-intercept of the line that is tangent to the graph of \(y = f(x)\) at the point \((x_1, f(x_1))\). This is depicted in Figure 1. A continuation of this process yields N.M. (2).
2.2 A simplified Newton’s method (S.N.M.)

The simplified Newton’s method (S.N.M.) given in [10] is similar to N.M. except that we keep $f'(x_0)$ fixed in (2):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}, \quad n = 0, 1, 2, \ldots$$

(4)

To derive the recurrence formula (4), we begin as we did for N.M. by linearizing $f$ in (1) about $x = x_0$. Then $x_1$ is the solution of the linearized equation (3),

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

if and only if $x_1$ is the $x$-intercept of the line that is tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$.

Unlike in the derivation of N.M., however, we do not repeat the process by linearizing $f$ about $x = x_1$ next; instead, we translate the first tangent line in a parallel fashion so that it intersects the graph of $y = f(x)$ at the point $(x_1, f(x_1))$, and let $x_2$ be the $x$-intercept of the translated tangent line:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_0)}.$$

This is depicted in Figure 2. Next, we translate the first tangent line in a parallel fashion so that it intersects the graph of $y = f(x)$ at the point $(x_2, f(x_2))$, and
let $x_3$ be the $x$-intercept of the new translated tangent line:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_0)}.$$  

A continuation of this process of parallel translations of the first tangent line leads to the S.N.M. (4).

2.3 Halley’s method (H.M.)

Another numerical root-finding method is Halley’s method (H.M.)$^2$:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}, \quad n = 0, 1, 2, \ldots$$  

(5)

To derive the recurrence formula (5), we first replace $f$ in (1) by its 2nd-degree Taylor polynomial centered at $x = x_0$. This gives

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 = 0.$$  

(6)

Next, we let $x_1$ be a solution of (6), factorize $x_1 - x_0$ from the last two terms on the left-hand side, and isolate the outside factor of $x_1 - x_0$. This yields

$$x_1 - x_0 = \frac{-f(x_0)}{f'(x_0) + \frac{f''(x_0)}{2}(x_1 - x_0)}.$$  

(7)

Then, we use N.M. (2) to approximate the factor $x_1 - x_0$ on the right-hand side of (7) to obtain, after some algebra,

$$x_1 = x_0 - \frac{2f(x_0)f'(x_0)}{2f'(x_0)^2 - f(x_0)f''(x_0)}.$$  

A repetition of this process successively with $x_1, x_2, \ldots$ leads to H.M. (5).

(Frame [4] was the first to derive H.M. using a 2nd-degree Taylor polynomial. We can also get H.M. by applying N.M. to the function $F = f/\sqrt{f}$, a fact that was first pointed out by Bateman [1].)

Surprisingly, H.M. also sports a geometrical interpretation, namely, each approximation $x_{n+1}$ is the $x$-intercept of the hyperbola that is osculatory (2nd-order-tangent) to the graph of $y = f(x)$ at the point $(x_n, f(x_n))$; see Figure 3. (Salehov [8] was apparently the first to suggest this.) To see this, consider the hyperbola

$$h(x) = \frac{(x - x_n) + e}{a(x - x_n) + b},$$

$^2$Yes, this is the Halley of “Halley’s comet” fame. See [9] for some historical background.
and solve the system of equations

\[ h(x_n) = f(x_n), \quad h'(x_n) = f'(x_n), \quad h''(x_n) = f''(x_n) \]

for the coefficients \(a, b, c\). Then, the fact that \(x_{n+1} = x_n - c\) if \(h(x_{n+1}) = 0\) leads to H.M. (5). Details of this derivation can be found in [9].

3 Comparing the methods

A rigorous treatment of convergence and the speed of convergence of numerical root-finding methods is beyond the scope of this note; on these matters, the interested reader may consult a numerical analysis textbook, for example, [6, 11]. Instead, we illustrate with a numerical example\(^3\)—using each method to approximate \(\sqrt{2} = 1.414213562\ldots\)—what is suggested by the geometry that is depicted in Figures 1–3: H.M. “beats” N.M. which, in turn, “beats” the S.N.M. (This is, in fact, generally the case.)

\(^3\)Numerical results are obtained using Maple V Release 4.
Table 1: Approximating $\sqrt{2} = 1.414213562\ldots$

<table>
<thead>
<tr>
<th>$x_0 = 2$</th>
<th>N.M.</th>
<th>S.N.M.</th>
<th>H.M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1.500000000</td>
<td>1.500000000</td>
<td>1.428571429</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1.416666667</td>
<td>1.437500000</td>
<td>1.414213927</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1.414215686</td>
<td>1.420898438</td>
<td>1.414215362</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1.414213562</td>
<td>1.41610345</td>
<td>1.414213562</td>
</tr>
<tr>
<td>$x_5$</td>
<td>1.414213562</td>
<td>1.414782814</td>
<td>1.414213562</td>
</tr>
<tr>
<td>$x_6$</td>
<td>1.414213562</td>
<td>1.414380211</td>
<td>1.414213562</td>
</tr>
</tbody>
</table>

Toward this end, we let $f(x) = x^2 - 2$ and $x_0 = 2$. We find that for

N.M. \[ x_{n+1} = \frac{x_n^2 + 2}{2x_n}, \quad n = 0, 1, 2, \ldots, \]

S.N.M. \[ x_{n+1} = \frac{-x_n^2 + 2x_0x_n + 2}{2x_0}, \quad n = 0, 1, 2, \ldots, \]

H.M. \[ x_{n+1} = \frac{x_n^3 + 6x_n}{3x_n^2 + 2}, \quad n = 0, 1, 2, \ldots. \]

Table 1 shows that H.M. approximates $\sqrt{2}$ to eight decimal places (d.p.) in $x_3$, and N.M. in $x_4$; the S.N.M. does not approximate $\sqrt{2}$ to eight d.p. even by $x_6$. (In fact, we must wait until $x_{16} = 1.414213563$ before the S.N.M. approximates $\sqrt{2}$ to eight d.p.)

4 A dynamical systems approach

In this section, we take a dynamical systems approach to N.M., the S.N.M., and H.M. From this point of view, we provide a graphical analysis of the convergence of the methods in particular examples, as well as a look at a couple of ways in which the methods may fail.

4.1 Preliminaries

The orbit of a point $x_0$ under iteration of a function $\phi$ is the sequence of points given by the recurrence formula

\[ x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, \ldots. \]
Here, $\phi$ is called an iteration function (I.F.).

A point $\alpha$ is called a fixed point of $\phi$ if $\phi(\alpha) = \alpha$. A fixed point $\alpha$ is said to be attracting if there is an open interval $I$ containing $\alpha$ for which the orbit of $x$ converges to $\alpha$ for all $x \in I$; and $\alpha$ is said to be repelling if the orbit of $x$ tends away from $\alpha$ for all $x \in I$. See Figures 4(a)–(b).

An important basic result is that a fixed point $\alpha$ is attracting if $|\phi'(\alpha)| < 1$, and it is repelling if $|\phi'(\alpha)| > 1$. If $\phi^n(\alpha) = \alpha$—where, for example, $\phi^2(\alpha) = (\phi \circ \phi)(\alpha)$—then we call $\alpha$ a periodic point of period $n$; the least positive $n$ for which $\phi^n(\alpha) = \alpha$ is called the prime period of $\alpha$. If, for some $m > 0$, $\phi^i(\alpha)$ is periodic for $i \geq m$, then we call $\alpha$ an eventually periodic point.

For example, consider the function $\phi(x) = -x^3$. Then, $\phi(0) = 0$ and $\phi'(0) = 0$ implies that zero is an attracting fixed point of $\phi$. Note that $\phi$ has periodic orbits of period two at $\pm 1$. See Figure 5.\(^4\)

(For a thorough introduction to dynamical systems, see [3].)

\(^4\)Orbits are computed and plotted using Mary Ann Software PSMathGraphsH.
4.2 Connection with N.M., the S.N.M., and H.M.

Given a function $f$, define another function, $N_f$, by

$$N_f(x) = x - \frac{f(x)}{f'(x)},$$  

(8)

that we call the Newton I.F. for $f$. Observe that the orbit of a point $x_0$ under iteration of $N_f$ is precisely the sequence of points we would obtain by applying N.M. (2) to an initial guess $x_0$. Further, $\alpha$ is a simple root of $f(x) = 0$ if and only if $\alpha$ is a fixed point of $N_f$. This shows us another way to study numerical root-finding methods like N.M., namely, as I.F. in dynamical systems. For the S.N.M. and H.M., we find that the simplified Newton I.F. and the Halley I.F. are, respectively,

$$SN_f(x; x_0) = x - \frac{f(x)}{f'(x_0)},$$  

(9)

$$H_f(x) = x - \frac{2f(x)f'(x)}{2f'(x)^2 - f(x)f''(x)},$$  

(10)

Note that $SN_f$ depends on a parameter, $x_0$, the initial guess for a root of $f(x) = 0$.

To illustrate, we apply (8)–(10) to the example of approximating $\sqrt{2}$ in
Section 3. First, recall that \( f(x) = x^2 - 2 \) and \( x_0 = 2 \). Then,

\[
N_f(\sqrt{2}) = \sqrt{2}, \quad N'_f(\sqrt{2}) = 0,
\]
\[
SN_f(\sqrt{2}; 2) = \sqrt{2}, \quad SN'_f(\sqrt{2}; 2) = 1 - \frac{\sqrt{2}}{2} < 1,
\]
\[
H_f(\sqrt{2}) = \sqrt{2}, \quad H'_f(\sqrt{2}) = 0
\]

implies that \( \sqrt{2} \) is an attracting fixed point in each case. Figures 6(a)–(d) show the orbit of 2 converging to the fixed point \( \sqrt{2} \) for the three I.F., which corresponds to the initial guess \( x_0 = 2 \) converging to \( \sqrt{2} \), the positive root of \( x^2 - 2 = 0 \), using N.M., the S.N.M., and H.M.

4.3 A second comparison

Garrett’s Web page [5] provides an interactive animation of N.M. applied to the function \( f(x) = x^3 - 3x^2 + 2x + 0.4 \). The animation [5] demonstrates, using tangent lines, that “the close approach of the graph to the \( x \)-axis in the ‘trough’ [local minimum] causes bizarre and unpredictable [sic] ‘deflections’ from the true root.” Here, we undertake a graphical analysis of the Newton I.F. for the function \( f \). Note that the equation \( f(x) = 0 \) has exactly one (simple) real root, which is approximately \(-0.159704853\).

Figure 7(a) shows the graph of \( y = f(x) \) (heavy line)—which has a local maximum at \( 1 - \sqrt{3}/3 \) and a local minimum at \( 1 + \sqrt{3}/3 \)—and the graph of \( y = N_f(x) \)—which has vertical asymptotes at the critical points of \( f \). Figure 7(b) shows the orbits of 0.25 and 1 under iteration of \( N_f \). We see that the orbit of 0.25 converges to the root of \( f(x) = 0 \). On the other hand, the orbit of 1 appears erratic—which stems from the fact that the orbit jumps back and forth between the branches of the graph of \( y = N_f(x) \) on either side of the vertical asymptote \( 1 + \sqrt{3}/3 \)—and appears not to converge at all. Figure 7(c) shows, however, that eventually the orbit of 1 does converge to the root of \( f(x) = 0 \).

What is the orbit of 1 under iteration of

\[
SN_f(x; 1) = x^3 - 3x^2 + 3x + 0.4,
\]
\[
H_f(x) = \frac{15x^5 - 45x^4 + 35x^3 - 12x^2 + 18x - 4}{30x^4 - 120x^3 + 165x^2 - 96x + 26}
\]

We consider \( H_f \) first. We note that \( H_f \) has three fixed points, one each at the simple zero and the two critical points of \( f \). In addition, \( H_f \) has two vertical
(a) $N_f(x) = (x^2 + 2)/(2x)$. 

(b) $SN_f(x; 2) = (-x^2 + 4x + 2)/4$. 

(c) $H_f(x) = (x^3 + 6x)/(3x^2 + 2)$. 

(d) A blow-up of $[1.35, 1.5] \times [1.35, 1.5]$ in Fig. 6(c).

Figure 6: The orbit of 2 under iteration of $N_f$, $SN_f(\cdot; 2)$, and $H_f$, where $f(x) = x^2 - 2$, converging to the fixed point $\alpha = \sqrt{2}$. 

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(a) The graphs of \( y = f(x) \) (heavy line) and \( y = N_f(x) \).

(b) The orbits of 0.25 and 1.

(c) The orbit of 1 after 90 iterations.

Figure 7: A graphical analysis of N.M. applied to \( f(x) = x^3 - 3x^2 + 2x + 0.4 \) by examining its Newton I.F., \( N_f(x) = (10x^3 - 15x^2 - 2)/(15x^2 - 30x + 10) \).
asymptotes, at about 1.523 802 619 and 1.631 467 004. As a result, Figure 8 shows that the orbit of 1 under iteration of \( H_f \) looks to be even more complicated than that under iteration of \( N_f \). Figure 8(c) shows that eventually, however, the orbit of 1 converges to the root of \( f(x) = 0 \).

Next, we consider \( SN_f(x; 1) \), which has exactly one fixed point at the root of \( f(x) = 0 \). Figure 9(a) shows, however, that the orbit of 1 does not converge to the fixed point; rather, the orbit “escapes to infinity.” In fact, Figure 9(b) shows that the orbit of 1 tends monotonely toward \( 1 + \sqrt{3}/3 \), the critical point of the local minimum of \( f \), but then continues past that value. Hence, we see that the S.N.M. fails to approximate a root of \( f(x) = 0 \) in this case.

### 4.4 Failure of the method

We discuss a few ways in which N.M. and the S.N.M. may fail. For the functions that we would usually consider, H.M. is globally convergent [2].

#### 4.4.1 Escape to infinity

As a first example, consider the equation \( \arctan(x - 1) = 0 \) that has exactly one root, namely, 1. Let \( f(x) = \arctan(x - 1) \). Then, the Newton I.F. for \( f \) is

\[
N_f(x) = x - (1 + (x - 1)^2) \arctan(x - 1),
\]

and a straightforward calculation shows that \( N'_f(1) = 0 \) so that 1 is an attracting fixed point of \( N_f \). Figure 10(a) shows that 2 converges to 1 under iteration of \( N_f \), so an initial guess of \( x_0 = 2 \) converges to 1 using N.M. On the other hand, Figure 10(b) shows that 2.5 does not converge under iteration of \( N_f \), so an initial guess of \( x_0 = 2.5 \) does not converge using N.M. (it escapes to infinity).

#### 4.4.2 Periodic points

As a second example, consider the function \( f \) given by

\[
f(x) = \begin{cases} 
\sqrt{x} & \text{if } x \geq 0, \\
-\sqrt{-x} & \text{if } x < 0.
\end{cases}
\]
Figure 8: A graphical analysis of H.M. applied to $f(x) = x^3 - 3x^2 + 2x + 0.4$ by examining its Halley I.F. In (c), we see the orbit of 1 jumping among three branches of the graph of $y = H_f(x)$. 
The orbit of 1 under iteration of $SN_f(x;1)$.

A blow-up of $[1.3, 1.8] 	imes [1.3, 1.8]$ in Fig. 9(a).

Figure 9: A graphical analysis of the S.N.M. applied to $f(x) = x^3 - 3x^2 + 2x + 0.4$ by examining its simplified Newton I.F.

An attracting orbit of $N_f$.

A repelling orbit of $N_f$.

Figure 10: In (a), the orbit of 2 converges to 1 under iteration of $N_f$; in (b), the orbit of 2.5 does not converge under iteration of $N_f$. 

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Figure 11: Every point $x_0$ is a periodic point of period two for $N_f$ as well as $SN_f(\cdot; x_0)$ in this example.

Then, zero is a root of $f(x) = 0$ as well as a fixed point of

\[
N_f(x) = -x,
\]

\[
SN_f(x; 2) = \begin{cases} 
  x - 2\sqrt{2x} & \text{if } x \geq 0, \\
  x + 2\sqrt{-2x} & \text{if } x < 0.
\end{cases}
\]

In both cases, we find that 2 is a periodic point of period two, so that 2 does not converge to the root zero of $f(x) = 0$ using either N.M. or the S.N.M.; see Figure 11. Indeed, in this particular example, every point $x_0$ is a periodic point of period two for $N_f$ as well as for $SN_f(\cdot; x_0)$.

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References


