The Normals to a Parabola and the Real Roots of a Cubic

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Around 200 b.c. Apollonius of Perga wrote Conics, a collection of books that includes results on normals to curves. In 1545 Geronimo Cardano produced Ars magna, a major work that contains the solution of the cubic equation. In 1659 John Wallis published Tractatus duo, prior de cycloide, posterior de cissoid (Two Treatises, the First on the Cycloid, the Second on the Cissoid), which has the first rectification of an algebraic curve, a result that was achieved by his student William Neil. We will see that these three events, separated by centuries in time, are connected by a thread in the fabric of mathematics.

Apollonius (ca. 262–190 b.c.) is best known for his extensive study of conic sections, Conics [1], [2], a collection of eight books. Heath [8, p. 158] says of this work that “Book V is of an entirely different order, indeed it is the most remarkable of the extant Books.” In Book V Apollonius investigates, for a given conic $C$ and a point $P$ not on $C$, the number of points $Q$ on $C$ for which the distance to $P$ is a local maximum or minimum. In Propositions 27–30 of Book V, he shows that such an extreme line passing through $P$ and $Q$ is perpendicular to the tangent of $C$ at $Q$. In modern terms, we say that these extreme lines are normals to the curve.

This note is on the number of normals to the parabola $y = x^2$. We will see that the geometric problem of finding the number of normals to the parabola through a given point is equivalent to the algebraic problem of finding the
number of distinct real roots of a cubic.¹ (We mean the number of distinct real roots throughout this note.) For this reason, it is remarkable that Apollonius had obtained precise results using purely synthetic geometry [8, pp. 163–166] almost 1700 years before Italian mathematicians solved the cubic equation [3]. We will also see that the algebraic curve that Neil (1637–1670) rectified [3, pp. 379 and 383], and that has come to bear his name, bridges the geometric and the algebraic problems.

The equivalence

A normal to the parabola \( y = x^2 \) through the point \((\alpha, \beta)\) satisfies the relation \((y - \beta)/(x - \alpha) = -1/(2x)\). This relation simplifies to

\[
x^3 + px + q = 0,
\]
where

\[
p = \frac{1 - 2\beta}{2} \quad \text{and} \quad q = \frac{-\alpha}{2}.
\]

Thus, we see that the number of normals to \( y = x^2 \) through \((\alpha, \beta)\) equals the number of real roots of the cubic equation (1).

Conversely, any cubic equation \( x^3 + bx^2 + cx + d = 0 \) can be transformed to the form (1) by the horizontal translation \( x \mapsto x - b/3 \). This translation obviously does not change the number of real roots of the equation. Thus, using (2), we see that the number of real roots of a cubic equation equals the number of normals to the parabola through the point \((\alpha, \beta) = (-2q, (1 - 2p)/2)\).

The discriminant and Neil’s parabola

One way to find the number of normals to a curve through a given point is to draw all the normals to the curve through the point. However, it is not always easy to determine the normals to a curve just by “eyeballing.” For instance, Figure 1 shows what we might expect to be the normals to \( y = x^2 \) through

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¹McGiffert [10] treats the number of normals to a parabola in a similar way. However, he does not point out explicitly the equivalence between finding the number of normals to a parabola and determining the number of real roots of a cubic. Also, [10] appeared in 1933, long before the advent of computers and software that permit students to experiment with instant results.
various points \((\alpha, \beta)\) just by appearances, and Figure 2 shows the actual normals to the parabola obtained by solving (1) and (2). So, because we are interested only in the \textit{number} of normals to the parabola, we need another method if we are to avoid having to solve a cubic.

Figure 1. What appear to be normals.

Figure 2. Actual normals.

Now, every intermediate algebra student learns that the number of real roots of the quadratic equation \(x^2 + bx + c = 0\) is determined by its discriminant, namely, \(b^2 - 4c\). It is similarly easy to tell the number of real roots of (1), and the key is (surprise!) its discriminant, namely,

\[ D = \frac{q^2}{4} + \frac{p^3}{27}. \]
**Theorem 1.** Let \( n \) be the number of distinct real roots of the cubic equation \( x^3 + px + q = 0 \). If \( p = q = 0 \), then \( n = 1 \). Otherwise,

\[
n = 1 \text{ if } D > 0, \quad n = 2 \text{ if } D = 0, \quad \text{and} \quad n = 3 \text{ if } D < 0.
\]

(For a proof, see [6] or [11].) Note that \( D = 0 \) is a boundary across which the number of real roots changes.

Theorem 1 is an algebraic result (number of real roots) that was used to solve a geometric problem (number of normals). We would like a geometric result to solve the geometric problem. The key here is the graph of the function

\[
N(x) = \frac{3}{2^{4/3}} x^{2/3} + \frac{1}{2}. \tag{3}
\]

(The graph of the function \( y = x^n \), where \( n \) is a positive integer, is called a parabola of order \( n \) [11]. The graph of \( y = x^2 \) is also called the standard parabola and the graph of \( y = x^3 \) is also called the cubic parabola. The graph of \( y = N(x) \), therefore, is a semicubical parabola. As we mentioned, Neil was the first to rectify an algebraic curve [3, p. 378]. The curve Neil rectified was \( ay^2 = x^3 \) and, so, the semicubical parabola is often called Neil’s (or Neile’s) parabola [9], [11].)

Setting the discriminant \( D = 0 \) and using (2), we obtain \( \beta = N(\alpha) \). The following result then follows readily from Theorem 1.

**Theorem 2.** Let \( n \) be the number of normals to the parabola \( y = x^2 \) through the point \((\alpha, \beta)\). If \((\alpha, \beta) = (0, 1/2)\), then \( n = 1 \). Otherwise,

\[
n = 1 \text{ if } \beta < N(\alpha), \quad n = 2 \text{ if } \beta = N(\alpha), \quad \text{and} \quad n = 3 \text{ if } \beta > N(\alpha).
\]

Theorem 2 is a geometric result that solves the geometric problem: The number of normals to \( y = x^2 \) through a given point depends on whether the point is below, on, or above Neil’s parabola (3). Figure 3 shows the normals to the standard parabola through the points we looked at in Figure 2 together with Neil’s parabola (heavy curve). Neil’s parabola, therefore, is a geometric analogue of when the algebraic discriminant \( D = 0 \): It is a boundary across which the number of normals to the standard parabola changes just as the discriminant \( D = 0 \) is a boundary across which the number of real roots of a cubic equation changes.
The envelope of the family of normals

But what if we draw a whole lot of normals to $y = x^2$? Is a mess all that we get? Hardly! Indeed, Figure 4 shows that the envelope of the family of normals to the standard parabola resembles Neil’s parabola. Moreover, it appears that the number of normals to $y = x^2$ through a given point is consistent with Theorem 2 on both sides of the envelope.

The envelope of the family of normals to a curve is called the evolute of the curve. An evolute is also the locus of the centers of curvature of the curve [7]. For a curve that is parametrized by $(f(t), g(t))$, the evolute is given by

$$
\left( f - \frac{(f'^2 + g'^2)g'}{f'g'' - f''g'}, g + \frac{(f'^2 + g'^2)f'}{f'g'' - f''g'} \right).
$$

For the standard parabola $y = x^2$, a straightforward computation shows that

Figure 3. Normals to $y = x^2$ together with Neil’s parabola (heavy curve).

Figure 4. The family of normals to $y = x^2$ (with 40 normals shown).
the evolute is Neil’s parabola. Hence, the envelope of normals that appears in Figure 4 is, in fact, Neil’s parabola in Theorem 2. Consequently, we see that a line that is normal to the standard parabola at the point \((a, a^2)\) is tangent to Neil’s parabola at \((-4a^3, 3a^2 + 1/2)\), and conversely.

**The normals to** \(y = x^n\)

We now turn to curves that are the graphs of \(y = x^n\) for any integer \(n \neq 0\) or 1. A normal to \(y = x^n\) through the point \((\alpha, \beta)\) satisfies the relation \((y - \beta)/(x - \alpha) = -1/(nx^{n-1})\). This relation simplifies to \(r(x) = 0\), where

\[
r(x) = x^{2n-1} - \beta x^{n-1} + \frac{1}{n}x - \frac{\alpha}{n}.
\]

Thus, we see that the number of normals to \(y = x^n\) through \((\alpha, \beta)\) equals the number of real roots of \(r(x) = 0\). We call \(r\) the *resolvent* of \(y = x^n\).

With the aid of computer software, we can graph \(y = x^n\), its evolute (4), and its resolvent (5) for various integer exponents \(n\). Doing so sheds light on the number of normals to the curve through a given point for different values of \(n\). We find that there may be as many as five normals to \(y = x^n\) through some points. See Figure 5. (It turns out that there can never be more than five normals to \(y = x^n\) through some points. See Figure 5. (It turns out that there can never be more than five normals to \(y = x^n\). For if a polynomial has six or more distinct real zeros, Rolle’s theorem implies that its first derivative has at least five and its second derivative has at least four. But, for positive \(n\), we find that \(r''\) has at most three distinct real zeros. For negative \(n\), let \(k = -n\) and write \(r(x) = p(x)/x^{2k+1}\). Then we find that \(p'\) has at most four distinct real zeros.)

Students can continue to explore these ideas using computer software. Our computations and graphs were made using Octave [5], a free software similar to Matlab, and de Alwis [4] provides Mathematica code to generate the normals to a parabola from a given point. With the aid of such software, students can explore, for example, if there is a bound on the number of normals to \(y = |x|^{\gamma}\) through a given point for any real exponent \(\gamma\).

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Figure 5. The graphs of $y = x^n$, its evolute (dashed curve), and its resolvent (heavy curve) for the point $(1/10, 1)$ that is marked as $\times$.

References


